TAU FUNCTION AND CHERN-SIMONS INVARIANT

ANDREW MCINTYRE AND JINSUNG PARK

Abstract. We define a Chern-Simons invariant for a certain class of infinite volume hyperbolic 3-manifolds. We then prove an expression relating the Bergman tau function on a cover of the Hurwitz space, to the lifting of the function F defined by Zograf on Teichmüller space, and another holomorphic function on the cover of the Hurwitz space which we introduce. If the point in cover of the Hurwitz space corresponds to a Riemann surface X, then this function is constructed from the renormalized volume and our Chern-Simons invariant for the bounding 3-manifold of X given by Schottky uniformization, together with a regularized Polyakov integral relating determinants of Laplacians on X in the hyperbolic and singular flat metrics. Combining this with a result of Kokotov and Korotkin, we obtain a similar expression for the isomonodromic tau function of Dubrovin. We also obtain a relation between the Chern-Simons invariant and the eta invariant of the bounding 3-manifold, with defect given by the phase of the Bergman tau function of X.

1. Introduction

Let $\mathfrak{M}_g$ be the moduli space of compact Riemann surfaces of genus g, and let $\mathfrak{T}_g$ be the corresponding Teichmüller space of marked surfaces. Let $H_{g,n}(k_1,\ldots,k_\ell)$ be the Hurwitz space of equivalence classes $[\lambda : X \to \mathbb{CP}^1]$ of degree n holomorphic maps from a compact Riemann surface X to the Riemann sphere with ramification index $(k_1,\ldots,k_\ell)$ at infinity, and all ramification points being simple. Equipping X with a marking—a choice of standard generators of $\pi_1(X)$—gives a covering space $\tilde{H}_{g,n}(k_1,\ldots,k_\ell)$, in the same way that one obtains the covering $\mathfrak{T}_g$ of $\mathfrak{M}_g$. We will also be concerned with a space $\mathcal{H}_g(k_1,\ldots,k_m)$, whose fiber over a point in $\mathfrak{M}_g$ is the space of holomorphic 1-forms on the corresponding Riemann surface with zeroes of order $k_1,\ldots,k_m$, and we write $\tilde{\mathcal{H}}_g(k_1,\ldots,k_m)$ for the corresponding fiber space over $\mathfrak{T}_g$. (See Section 2 for precise definitions.)

In [7], Kokotov and Korotkin introduced the object τ_B, referred to as the Bergman tau function, with the property that τ_B^{24} is a globally well-defined holomorphic function on $\tilde{H}_{g,n}(k_1,\ldots,k_\ell)$. In [10], they defined τ_B in the same way for $\tilde{\mathcal{H}}_g(k_1,\ldots,k_m)$, such that τ_B^{24} is a globally well-defined holomorphic function on $\tilde{\mathcal{H}}_g(k_1,\ldots,k_\ell)$.

The main result of this paper is the following theorem.

Theorem 1.1. *Over $\tilde{H}_{g,n}(1,\ldots,1)$, $g \geq 1$, we have the following equality:*

$$\tau_B^{24} = c\,\exp\left(4\pi\mathbb{CS} + \frac{1}{\pi}I\right)F^{24}. \tag{1.1}$$

The same equality holds for the function τ_B^{24} on $\tilde{\mathcal{H}}_g(1,\ldots,1)$, $g \geq 1$.

Here c represents a constant, depending on g, n, and a topological choice that will be explained in Section 8. The complex-valued function $\mathbb{CS}$ on $\tilde{H}_{g,n}(1,\ldots,1)$ or $\tilde{\mathcal{H}}_g(1,\ldots,1)$ is

Date: September 20, 2012.
2010 Mathematics Subject Classification 58J28, 58J52, 37K10, 32G15, 14J15.

defined as follows. Each marked compact Riemann surface X has a Schottky uniformization given by a unique marked normalized Schottky group Γ; the group naturally defines an infinite volume hyperbolic 3-manifold M_X whose conformal boundary is X. A 1-form Ψ on X (here Ψ is either $d\lambda$ for the meromorphic function λ, or it is the holomorphic 1-form Φ) determines a singular framing on X, and there exists a singular framing s_Ψ on M_X which extends the framing on X in a sense we prescribe. In Section 4 we define an invariant $\mathbb{CS}(M,s)$ for a certain class of 3-manifolds M and singular framings s on M. The value of $\mathbb{CS}$ at a point corresponding to (X,Ψ) is then defined to equal $\mathbb{CS}(M_X, s_\Psi)$. Our definition of $\mathbb{CS}(M,s)$ is motivated by the work of Meyerhoff [16] and Yoshida [18] for finite volume hyperbolic 3-manifolds with cusps. In subsection 4.5 we show

$$\mathbb{CS}(M,s) = \frac{1}{\pi^2} W(M) + 2iCS(M,s), \tag{1.2}$$

where $W(M)$ is the renormalized volume of M (see [12], [17], [13]; we use the definition of Section 8 of [13]), and $CS(M,s)$ is the integral of the usual Chern-Simons 3-form over M with the framing s, together with a correction term corresponding to the singularities of the framing. Let us remark that $CS(M,s)$ is finite by our construction without any renormalization process and is well defined only up to $\frac{1}{2}\mathbb{Z}$.

The function I is real-valued, and is given by an explicit integral over the Riemann surface, involving the 1-form Ψ. We refer to I as a regularized Polyakov integral, since it plays the role of the usual Polyakov integral in relating the determinant of the Laplacian in the hyperbolic metric on X to that in the flat singular metric on X defined by Ψ, as we show in Corollary 1.4. Its precise definition is given in (6.1) and (9.4). The combined expression $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)$ gives a holomorphic function over $\tilde{H}_{g,n}(1,\dots,1)$ or $\tilde{\mathcal{H}}_g(1,\dots,1)$ (although by itself, I is actually a function over $H_{g,n}(1,\dots,1)$ or $\mathcal{H}_g(1,\dots,1)$). The function F is the holomorphic function over $\mathfrak{T}_g$ defined by Zograf in [19] (it is related to determinants of Laplacians—see below).

Theorem 1.1 allows us to interpret the Bergman tau function as a higher genus generalization of the Dedekind eta function. When $g=1$, it is known that $\tau_B = \eta(\tau)^2$ and $F = \prod_{m=1}^{\infty}(1-q^m)^2$ on $\mathcal{H}_1 \simeq \mathfrak{T}_1 \times \mathbb{C}^\times$ where $q = e^{2\pi i \tau}$, $\tau \in H^2 \simeq \mathfrak{T}_1$, and by elementary computation we have $\mathbb{CS} = i\tau$ and $I = 0$. Consequently in this case, Theorem 1.1 reduces to the 48-th power of the defining equation of the Dedekind eta function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1-q^m).$$

In [9], [10], it was shown that τ_B^{24} satisfies a modular property with respect to the mapping class group, which reduces to the modular property of η^{48} in genus 1. Further, the function F was shown in [20] (see also [15]) to have an infinite product expansion on a subset of $\mathfrak{T}_g$:

$$F = \prod_{\{\gamma\}} \prod_{m=1}^{\infty} (1-q_\gamma^m). \tag{1.3}$$

Here the first product runs over all primitive closed geodesics γ in M_X, and the complex number q_γ has modulus $e^{-\mathrm{length}(\gamma)}$ and argument given by the holonomy around γ in an orthogonal plane. The equation (1.3) is valid whenever the exponent of convergence δ of Γ is strictly less than 1.

The relation between objects on the 2-manifold X and the bounding infinite volume 3-manifold M_X given by Theorem 1.1 fits well with principle of "holography"—for example, see [14] and [17]. In this context, the Schottky uniformization provides a natural choice of bounding 3-manifold M_X.

In [8], Kokotov and Korotkin showed that the Bergman tau function τ_B is related to the isomonodromic tau function τ_I for $\tilde{H}_{g,n}(k_1,\ldots,k_\ell)$ considered as an underlying space of a Frobenius manifold in the sense of Dubrovin in [3], [4], by the equation $\tau_B = \tau_I^{-2}$. This implies the corollary

Corollary 1.2. *Over $\tilde{H}_{g,n}(1,\ldots,1)$, $g \geq 1$, we have the following equality:*

$$\tau_I^{48} = c\,\exp\Big(-4\pi\mathbb{CS} - \frac{1}{\pi}I\Big)\,F^{-24}. \tag{1.4}$$

Here and below, as in Theorem 1.1, c represents a constant depending on g, n, and possibly a topological choice. However, it does not always represent the same constant.

To state the second corollary of Theorem 1.1, we need a result about the phase of F. In [6], it is shown that the eta invariant of the odd signature operator over M_X is well-defined, without any additional renormalization, and it is proved that the phase of F at X is $\exp(-\frac{\pi i}{2}\eta(M_X))$, whenever the marked Schottky group Γ has exponent of convergence $\delta < 1$. We refer to [6] for more details. Combining this with (1.1), we have

Corollary 1.3. *The following equality holds*

$$\exp\Big(8\pi iCS - 12\pi i\eta\Big) = c\,\left(\frac{\tau_B}{|\tau_B|}\right)^{24}$$

over the subset of $\tilde{H}_{g,n}(1,\ldots,1)$ or $\tilde{\mathcal{H}}_g(1,\ldots,1)$, $g \geq 1$, for which the corresponding marked Schottky group Γ has exponent of convergence $\delta < 1$.

Let us remark that $\exp(4\pi iCS(M)) = \exp(6\pi i\eta(M))$ for any closed 3-manifold M. Hence Corollary 1.3 generalizes this equality for Schottky hyperbolic 3-manifolds, where the boundary Riemann surface X produces a defect term given by the phase of τ_B.

The quantities in the main theorem are related to regularized determinants of Laplacians. In [19] (see also [15]), it was shown that

$$\frac{\det\Delta_{\text{hyp}}}{A_{\text{hyp}}\det\langle\Phi_j,\Phi_k\rangle} = c\exp\big(-\frac{1}{12\pi}S\big)|F|^2 \qquad \text{over} \quad \mathfrak{T}_g$$

where Δ_{hyp} is the Laplacian in the unique metric of constant curvature -1 on X, A_{hyp} is the area of X in that metric, $\{\Phi_1,\ldots,\Phi_g\}$ is a basis of holomorphic 1-forms normalized with respect to the marking, and S is the real valued classical Liouville action functional over $\mathfrak{T}_g$. Note that this is distinct from the usual expression of $\det\Delta_{\text{hyp}}$ in terms of the Selberg zeta function; in particular, F is holomorphic in moduli. It is known that $S(X) = -4W(M_X)$, when M_X is related to X as above (see [12], [17], [13]). In [10], Kokotov and Korotkin showed that

$$|\tau_B|^2 = c\frac{\det\Delta_{\text{flat}}}{A_{\text{flat}}\det\langle\Phi_j,\Phi_k\rangle} \qquad \text{over} \quad \tilde{\mathcal{H}}_g(1,\ldots,1) \tag{1.5}$$

where Δ_{flat} is the Laplacian in the flat (singular) metric defined by Φ, and A_{flat} is the area of X in that metric. Combining these, we have

$$|\tau_B|^{24} = c\exp\big(\frac{4}{\pi}W\big)\Big(\frac{\det\Delta_{\text{flat}}}{A_{\text{flat}}}\cdot\frac{A_{\text{hyp}}}{\det\Delta_{\text{hyp}}}\Big)^{12}|F|^{24} \qquad \text{over} \quad \tilde{\mathcal{H}}_g(1,\ldots,1). \tag{1.6}$$

Observing that τ_B^{24} and F^{24} in (1.6) are holomorphic functions over $\tilde{\mathcal{H}}_g(1,\dots,1)$, it is natural to expect that there might exist a holomorphic function over $\tilde{\mathcal{H}}_g(1,\dots,1)$ whose modulus is $\exp\big(\frac{4}{\pi}W\big)\big(\frac{\det\Delta_{\mathrm{flat}}}{A_{\mathrm{flat}}}\cdot\frac{A_{\mathrm{hyp}}}{\det\Delta_{\mathrm{hyp}}}\big)^{12}$. One motivation for this work was to find such a holomorphic function, and Theorem 1.1 gives an answer to this question. Combining Theorem 1.1 and (1.6), and using the fact that I descends to $\mathcal{H}_g(1,\dots,1)$, we have the following Polyakov formula,

Corollary 1.4.

$$\frac{\det\Delta_{\mathrm{flat}}}{A_{\mathrm{flat}}}\cdot\frac{A_{\mathrm{hyp}}}{\det\Delta_{\mathrm{hyp}}} = c\,\exp\big(\frac{1}{12\pi}I\big) \qquad \textit{over} \quad \mathcal{H}_g(1,\dots,1),\ g\geq 1.$$

Note that the usual argument proving the Polyakov formula for two smooth metrics does not apply in our case, since the domains of Δ_{flat} and Δ_{hyp} are different. Let us also remark that this formula can be proved combining the results in [7] and [10].

We have restricted attention to $\tilde{H}_{g,n}(1,\dots,1)$ and $\tilde{\mathcal{H}}_g(1,\dots,1)$ for simplicity, but we expect the results above will hold for other $\tilde{H}_{g,n}(k_1,\dots,k_\ell)$ and $\tilde{\mathcal{H}}_g(k_1,\dots,k_m)$, with only minor adjustments in the definitions of $\mathbb{CS}$ and I and slight changes in the proofs. We also note in passing that our constructions of $\mathbb{CS}(M,s)$ and $I(X,\Psi)$ can be extended in a straightforward way to apply when M is any convex co-compact hyperbolic 3-manifold with conformal boundary X. In this case we expect that our methods will show that $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)$ is locally a holomorphic function on the associated deformation space. This is a parallel of Yoshida's result in [18] for finite volume hyperbolic manifolds with cusps, where I is a new "defect" term coming from the boundary of genus $g>1$.

In Section 2, we give the necessary background and make precise definitions. In Sections 3 through 8, for simplicity of exposition, we present the proof of Theorem 1.1 over $\tilde{\mathcal{H}}_g(1,\dots,1)$ only. In the last Section, we describe the necessary modifications to establish Theorem 1.1 over $\tilde{H}_{g,n}(1,\dots,1)$.

Acknowledgments We are grateful to Leon Takhtajan for his helpful comments and questions for the early version of this paper. We are also thankful to Aleksey Kokotov and Dmitry Korotkin for useful discussions about their works on the Bergman tau function. The work of the second author is partially supported by the SRC-GaiA.

2. Preliminary background

2.1. **Hurwitz spaces and Tau functions.** Let $H_{g,n}(k_1,\dots,k_\ell)$ be the Hurwitz space of equivalence classes $[\lambda: X\to\mathbb{CP}^1]$ of n-fold branched coverings

$$\lambda: X\to\mathbb{CP}^1$$

where X is a compact Riemann surface of genus g and the holomorphic map λ of degree n satisfies the following conditions:

i) the map λ has m simple ramification points $p_1,\dots,p_m\in X$ with distinct finite images $\lambda_1,\dots,\lambda_m\in\mathbb{C}\subset\mathbb{CP}^1$,

ii) the preimage $\lambda^{-1}(\infty)$ consists of ℓ points: $\lambda^{-1}(\infty)=\{q_1,\dots,q_\ell\}$ and the ramification index of the map λ at the point q_j is k_j for $1\leq j\leq\ell$.

Here two branched coverings $\lambda: X\to\mathbb{CP}^1$ and $\lambda': X'\to\mathbb{CP}^1$ are equivalent if there exists a biholomorphic map $f: X\to X'$ such that $\lambda'\circ f=\lambda$. Note that $n=k_1+\cdots+k_\ell$ and $m=2g-2+n+\ell$ by the Riemann-Hurwitz formula. We also introduce the covering

$\hat{H}_{g,n}(k_1,\ldots,k_\ell)$ of the space $H_{g,n}(k_1,\ldots,k_\ell)$ consisting of pairs

$$\big(\ [\lambda : X \to \mathbb{CP}^1], \{a_i, b_i \mid 1 \le i \le g\}\ \big)$$

where $[\lambda : X \to \mathbb{CP}^1] \in H_{g,n}(k_1,\ldots,k_\ell)$ and $\{a_i, b_i \mid 1 \le i \le g\}$ denotes a Torelli marking on X, that is, a canonical basis of $H_1(X,\mathbb{Z})$. The space $\hat{H}_{g,n}(k_1,\ldots,k_\ell)$ is a connected complex manifold of dimension $m = 2g-2+n+\ell$, and the local coordinates on this manifold are given by the finite critical values of the map λ, that is, $\lambda_1,\ldots,\lambda_m$.

In [8], [10], the Bergman tau function τ_B over $\hat{H}_{g,n}(k_1,\ldots,k_\ell)$ is defined in terms of the Bergman kernel. The Bergman kernel on a Riemann surface X with a Torelli marking is defined by $B(p,q) := d_p d_q \log E(p,q)$ for $p,q \in X$ where $E(p,q)$ is the prime form on X. Near the diagonal $p = q$, the Bergman kernel $B(p,q)$ has the expression

$$B(z(p),z(q)) = \big(\,\big(z(p)-z(q)\big)^{-2} + H\big(z(p),z(q)\big)\,\big)dz(p)dz(q)$$

where $z(p), z(q)$ are local coordinates of points p,q in X, and the Bergman projective connection R_B is defined in a local coordinate by

$$R_B(z(p)) = 6\lim_{q\to p} H(z(p),z(q)). \tag{2.1}$$

The meromorphic function λ also defines a projective connection $R_{d\lambda}$, which is defined in a local coordinate to be $\mathcal{S}(\lambda)$, where $\mathcal{S}$ is the Schwarzian derivative defined by

$$\mathcal{S}(f) = \left(\frac{f_{zz}}{f_z}\right)_z - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

Now the Bergman tau function τ_B over $\hat{H}_{g,n}(k_1,\ldots,k_\ell)$ is locally defined to be a holomorphic solution of the system of compatible equations

$$\frac{\partial \log \tau_B}{\partial \lambda_i} = \frac{\sqrt{-1}}{12\pi}\int_{s_i} \frac{R_B - R_{d\lambda}}{\lambda_z}\,dz \qquad \text{for} \quad i = 1,\ldots,m,$$

where s_i is a small circle around the ramification point $p_i \in X$, in a local coordinate z near p_i. Note that the difference $R_B - R_{d\lambda}$ is a meromorphic quadratic differential and $\frac{R_B - R_{d\lambda}}{\lambda_z}\,dz$ is a meromorphic 1-form. It follows from [9] that τ_B^{24} is globally well-defined on $\hat{H}_{g,n}(k_1,\ldots,k_\ell)$.

The Bergman tau function τ_B is related to the isomonodromic tau function τ_I of Dubrovin [3], [4] by a theorem of Kokotov and Korotkin [8]:

Theorem 2.1.

$$\tau_B = \tau_I^{-2} \qquad \textit{over} \quad \hat{H}_{g,n}(k_1,\ldots,k_\ell).$$

Here $\hat{H}_{g,n}(k_1,\ldots,k_\ell)$ is considered as the underlying space of a Frobenius manifold where the isomonodromic tau function τ_I is defined; see [3], [4], [8] for details.

We also define the space $\mathcal{H}_g$ to be the moduli space of pairs (X,Φ) where X is a compact Riemann surface of genus $g > 1$ and Φ is a holomorphic 1-form over X. We denote by $\mathcal{H}_g(k_1,\ldots,k_m)$ the stratum of $\mathcal{H}_g$ consisting of differentials Φ which have m zeroes on X of multiplicities $(k_1,\ldots,k_m)$. For more details about these spaces, we refer to [11]. As before, we also introduce a covering $\hat{\mathcal{H}}_g(k_1,\ldots,k_m)$ of $\mathcal{H}_g(k_1,\ldots,k_m)$ consisting of triples $(X,\Phi,\{a_i,b_i \mid 1 \le i \le g\})$ where $\{a_i,b_i \mid 1 \le i \le g\}$ is a canonical basis of $H_1(X,\mathbb{Z})$.

Cutting the Riemann surface along the cycles given by a Torelli marking $\{a_i,b_i \mid 1 \le i \le g\}$, we get the fundamental polygon $\hat{X}$. Inside of $\hat{X}$ we choose $(m-1)$-paths l_j which connect the zero p_1 with the other zeros p_j for $j = 2,\ldots,m$. The set of paths a_i, b_i, l_j gives a basis in the relative homology group $H_1(X,(\Phi),\mathbb{Z})$ where $(\Phi)=\sum_{j=1}^m k_jp_j$ denotes the divisor of Φ. Following [10], local coordinates on $\hat{\mathcal{H}}_g(k_1,\dots,k_m)$ can be chosen as follows:

$$A_i:=\int_{a_i}\Phi,\qquad B_i:=\int_{b_i}\Phi,\qquad Z_j:=\int_{l_j}\Phi, \tag{2.2}$$

where $i=1,\dots,g$ and $j=1,\dots,m-1$. For simplicity, we also use another notation ζ_i for the coordinates defined by

$$\zeta_i:=A_i,\qquad \zeta_{g+i}:=B_i,\qquad \zeta_{2g+j}:=Z_{j+1}.$$

Define cycles s_i for $i=1,\dots,2g+m-1$ by

$$s_i=-b_i,\qquad s_{g+i}=a_i$$

for $i=1,\dots,g$ and define the cycle s_{2g+i} to be a small circle with positive orientation around p_{i+1}.

As before, Kokotov and Korotkin [10] also define the Bergman tau function τ_B over the stratum $\hat{\mathcal{H}}_g(k_1,\dots,k_m)$ to be a holomorphic solution of the following compatible system of equations:

$$\frac{\partial\log\tau_B}{\partial\zeta_i}=\frac{\sqrt{-1}}{12\pi}\int_{s_i}\frac{R_B-R_\Phi}{h}\,dz\qquad\text{for}\quad i=1,\dots,2g+m-1, \tag{2.3}$$

where $\Phi(z)=h(z)\,dz$ for a local coordinate z. Here R_B denotes the Bergman projective connection defined in (2.1) and R_Φ is the projective connection given by the Schwarzian derivative $\mathcal{S}(\int^z\Phi)$ with respect to a local coordinate z. It is shown in [10] that τ_B does not depend on the choice of the l_j, and that τ_B^{24} is a globally well-defined function on $\hat{\mathcal{H}}_g(k_1,\dots,k_m)$.

Finally we introduce covering spaces $\tilde{H}_{g,N}(k_1,\dots,k_\ell)$ and $\tilde{\mathcal{H}}_g(k_1,\dots,k_m)$ of $\hat{H}_{g,N}(k_1,\dots,k_\ell)$ and $\hat{\mathcal{H}}_g(k_1,\dots,k_m)$ respectively, by marking an ordered set of generators $\{a_i,b_i\mid 1\le i\le g\}$ of $\pi_1(X)$ rather than of $H_1(X,\mathbb{Z})$. There are canonical maps from these spaces to the Teichmüller space $\mathfrak{T}_g$ of marked Riemann surfaces of genus g. Note that the tau functions τ_I, τ_B can be lifted to these spaces. For simplicity we will mainly work over the spaces $\tilde{H}_{g,n}(1,\dots,1)$ and $\tilde{\mathcal{H}}_g(1,\dots,1)$ whose dimensions are $m=2g-2+2n$ and $4g-3$ respectively.

2.2. **Basic facts on Schottky groups and Schottky spaces.** Given a compact Riemann surface X of genus $g\ge 1$, there exists a Schottky uniformization of X, described as follows. A subgroup Γ of $PSL_2(\mathbb{C})$ is called a *Schottky group* if it is generated by $L_1,\dots,L_g$ satisfying the following condition: there exist $2g$ smooth Jordan curves C_r, $r=\pm1,\dots,\pm g$, which form the oriented boundary of a domain $D\subset\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ such that $L_rC_r=-C_{-r}$, $r=1,\dots,g$ where $PSL_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ in the usual way and the negative signs indicate opposite orientation. Any Schottky group gives a compact Riemann surface $X=\Gamma\backslash\Omega$ where $\Omega=\cup_{\gamma\in\Gamma}\gamma D$ is the set of discontinuity of the action of Γ on $\hat{\mathbb{C}}$, and every compact Riemann surface arises in this way. A Schottky group is *marked* if it is equipped with a particular choice of ordered set of free generators $L_1,\dots,L_g$. If the Riemann surface X is marked, then requiring the $b_1,\dots,b_g\in\pi_1(X)$ to map to $L_1,\dots,L_g$ fixes the marked Schottky group up to overall conjugation in $PSL_2(\mathbb{C})$.

We define a *Schottky 3-manifold* to be a smooth 3-manifold with boundary that is topologically a closed solid 3-dimensional handlebody $\overline{M}:=M\cup X$, where M is the corresponding open handlebody, and the boundary X is a compact smooth 2-dimensional surface. We call a Schottky 3-manifold *hyperbolic* if it is equipped with a complete hyperbolic metric g_M on M, and we call it *marked* if it is equipped with an ordered choice of generators of $\pi_1(M)$.

Any compact Riemann surface X with a uniformization by a marked Schottky group Γ gives a marked Schottky hyperbolic 3-manifold $M \cup X$ in the following way: $M = \Gamma \backslash H^3$ (where $PSL_2(\mathbb{C})$ acts on H^3 in the usual way), $X = \Gamma\backslash\Omega$, and the topology on $M \cup X$ is that inherited from $\overline{H^3} := H^3 \cup \hat{\mathbb{C}}$. The choice of the ordered set of generators $L_1, \dots, L_g$ gives the marking on $\pi_1(M)$, by identifying elements of Γ with deck transformations of the universal cover of $\overline{M}$. Conversely, by means of the developing map, every marked Schottky hyperbolic 3-manifold M arises from a marked Schottky group in this way, and the group is unique up to an overall conjugation in $PSL_2(\mathbb{C})$. When a marked Schottky group Γ and a marked Schottky hyperbolic 3-manifold $M \cup X$ correspond in this way, we will say that the group Γ *uniformizes* the manifold $\overline{M} = M \cup X$.

In summary, given a compact marked Riemann surface X, we obtain a unique marked Schottky hyperbolic 3-manifold $M \cup X$ whose conformal boundary is X. We will sometimes write $M = M_X$ if we want to emphasize that the manifold M is determined by the marked surface X.

For a fixed g, the *Schottky space of genus* g, denoted by $\mathfrak{S}_g$, is the set of all marked Schottky groups with g generators, modulo overall conjugation in $PSL_2(\mathbb{C})$. It is known that $\mathfrak{S}_g$ has a canonical complex manifold structure of dimension $3g-3$, and its universal cover is the Teichmüller space $\mathfrak{T}_g$, with the covering map being holomorphic. The generators L_i, $i = 1, \dots, g$, are holomorphic maps from $\mathfrak{S}_g$ to $PSL_2(\mathbb{C})$. In view of the uniformization discussed above, we implicitly identify $\mathfrak{S}_g$ with the deformation space of marked Schottky hyperbolic 3-manifolds.

Every Schottky hyperbolic 3-manifold is conformally compact: in some neighborhood $N \subset \overline{M}$ of X, there exists a smooth boundary defining function $r : N \to \mathbb{R}_{\geq 0}$ such that

i) $r > 0$ on $N \cap M$, $r = 0$ on X, and $dr = 0$ restricted to X,
ii) the rescaled metric $\overline{g} := r^2 g_M$ extends smoothly to $N \cap \overline{M}$,
iii) $|dr|^2_{\overline{g}} = 1$ in N.

We also write $\overline{g}$ for the extension of the metric $\overline{g}$ to $N \cap \overline{M}$. The conformal class of the metric $\overline{g}\big|_{TX}$ is independent of the choice of boundary defining function; hence the choice of a metric g_M induces a unique conformal class of metrics on the conformal boundary X. For genus $g > 1$, in each conformal class of metrics on X, there is a unique hyperbolic metric g_X of constant curvature -1. For genus $g = 1$, in each conformal class of metrics on X there is a unique flat metric g_X in which $\mathrm{Area}(X) = 1$. We will need a parametrization of a neighborhood $N \subset \overline{M}$ of the conformal boundary X. If we demand that $\overline{g}\big|_{TX}$ is equal to the metric g_X, then the boundary defining function satisfying the conditions above is unique. For a sufficiently small $a > 0$, this defining function r determines an identification of $X \times [0, a)$ with a subneighborhood $N_{[0,a)} \subset N$, by letting $(p, t) \in X \times [0, a)$ correspond to the point obtained by following the integral curve ϕ_t of $\nabla_{\bar{g}} r$ emanating from p for t units of time. Throughout the rest of the paper, we will fix such an a. For this defining function r, the t-coordinate is just r and $\nabla_{\bar{g}} r$ is orthogonal to the slices $X \times \{t\}$. Hence identifying t with r on $X \times [0, a)$, the hyperbolic metric g_M over M has the form

$$g_M = r^{-2}(g_r + dr^2)$$

over $N_{[0,a)}$, where g_r denotes a Riemannian metric over $X^r := X \times \{r\}$. See [5] for more details.

3. Framings over Schottky hyperbolic 3-manifolds

From here on, $\overline{M} = M \cup X$ will denote a marked Schottky hyperbolic 3-manifold with conformal boundary X. In this section, we define what we mean by a "singular framing" over M or over X, and we define a class of "admissible" singular framings which we will use to define the Chern-Simons invariant. We then describe how to assign, to each holomorphic 1-form Φ on X with only simple zeroes, an admissible singular framing on X. In Section 9 we will describe how to relax the assumptions on Φ. Finally, we prove that an admissible singular framing on X "extends" (in a sense to be defined below) to an admissible singular framing on M.

3.1. **Admissible singular framings.** Let $F(M)$ denote the $SO(3)$ frame bundle with the projection map $p : F(M) \to M$. For a subset $U \subset M$, by a *framing over* U we mean a section of $F(M)$ over U.

Let $\mathcal{L}$ denote an union of disjoint simple curves in M. A framing over $\mathcal{L}$ in M, written as $(e_1(y), e_2(y), e_3(y)) \in T_yM \oplus T_yM \oplus T_yM$ for each $y \in \mathcal{L}$, is called a *reference framing on* $\mathcal{L}$, if $e_1(y)$ is tangent to $\mathcal{L}$ at each $y \in \mathcal{L}$.

Let $\mathcal{N}^\epsilon(\mathcal{L})$ be an ϵ-neighborhood of $\mathcal{L}$ in the metric g_M. A choice of reference framing κ over $\mathcal{L}$ allows us to construct the *deleted* ϵ*-tube around* $\mathcal{L}$, which by definition we take to be a map

$$\alpha : (0, \epsilon) \times \mathcal{L} \times S^1 \to (\mathcal{N}^\epsilon(\mathcal{L})) \subset M,$$

constructed as follows: for each $(\rho, y, v) \in (0, \epsilon) \times \mathcal{L} \times S^1$, we take the unique geodesic starting at y with initial vector $\cos(v)e_2(y) + \sin(v)e_3(y)$, and travel a distance ρ from y to the point $\alpha(\rho, y, v)$.

Given a reference framing κ on $\mathcal{L}$, we define the corresponding *reference framing of the deleted* ϵ*-tube around* $\mathcal{L}$ by parallel translating the reference framing κ along the unique geodesic connecting y and $\alpha(\rho, y, v)$. This gives a lifting

$$\tilde{\alpha} : (0, \epsilon) \times \mathcal{L} \times S^1 \to p^{-1}(\mathcal{N}^\epsilon(\mathcal{L})) \subset F(M)$$

of the map α. The *standard cylinder over* $\mathcal{L}$ is the map

$$\psi : \mathcal{L} \times S^1 \to p^{-1}(\mathcal{L}) \subset F(M)$$

which takes the point $(y, v) \in \mathcal{L} \times S^1$ to the framing

$$\psi(y, v) := (e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y))$$

at the point y.

A matrix function

$$A : (0, \epsilon) \times \mathcal{L} \times S^1 \to SO(3)$$

acts on a framing $\tilde{\alpha}$ of the deleted ϵ-tube around $\mathcal{L}$ by fiberwise right multiplication:

$$(e_1, e_2, e_3) \cdot A(\rho, y, v) = (\sum_{i=1}^{3} e_i a_{i1}, \sum_{i=1}^{3} e_i a_{i2}, \sum_{i=1}^{3} e_i a_{i3}),$$

over a point $\alpha(\rho, y, v)$ where a_{ij} denotes (i, j)-entry of $A(\rho, y, v)$. We denote the resulting framing by $\tilde{\alpha} \cdot A$. A matrix function $A : \mathcal{L} \times S^1 \to SO(3)$ acts on the standard cylinder ψ to give $\psi \cdot A$ in the same fashion.

For a connected simple curve $\ell \subset M$, the *special singularity of index n at ℓ* is the framing $\tilde{\alpha} \cdot A_n$ over the deleted ϵ-tube around ℓ, where $\tilde{\alpha}$ is the reference framing on the deleted ϵ-tube around ℓ, and A_n is the matrix function on $(0, \epsilon) \times \ell \times S^1$ defined by

$$A_n(\rho, y, v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(nv) & -\sin(nv) \\ 0 & \sin(nv) & \cos(nv) \end{pmatrix}.$$

For fixed $y \in \ell$ and $v \in S^1$, the limit of $\tilde{\alpha} \cdot A_n$ as $\rho \to 0$ exists, and equals the framing $(e_1(y), \cos(nv)e_2(y) + \sin(nv)e_3(y), -\sin(nv)e_2(y) + \cos(nv)e_3(y))$ over y. Hence the map consisting of these limits as $\rho \to 0$ for all $y \in \ell$ and $v \in S^1$ is given by n-copies of the standard cylinder over ℓ. Here a negative integer n indicates opposite orientation. For $\mathcal{L}$ a disjoint union of simple curves, we say that a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$ has a *special singularity at* $\mathcal{L}$ if $\mathcal{F} \circ \alpha$ has the special singularity of index n for an integer n on each connected component of $(0, \epsilon) \times \mathcal{L} \times S^1$. Let us remark that n could be different over each component of $\mathcal{L}$. Our definition of special singularity coincides with Meyerhoff's [16] when $n = 1$.

For a connected simple curve $\ell \subset M$, the *admissible singularity of index n at ℓ* is the special singularity framing of index n at ℓ, acted on by a matrix function A:

$$\tilde{\alpha} \cdot A_n \cdot A : (0, \epsilon) \times \ell \times S^1 \to p^{-1}(\mathcal{N}^\epsilon(\ell)) \subset F(M), \tag{3.1}$$

where $A : (0, \epsilon) \times \ell \times S^1 \to SO(3)$ satisfies the condition that $\lim_{\rho \to 0} A(\rho, y, v)$ exists and is independent of v, for all $y \in \ell$ and $v \in S^1$. We say that a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$ has an *admissible singularity at* $\mathcal{L}$ if the limit of $\mathcal{F} \circ \alpha$ as $\rho \to 0$ exists for all $y \in \mathcal{L}$ and $v \in S^1$ and the map given by this limit is the same as the map given by the limit of $\tilde{\alpha} \cdot A_n \cdot A$ as $\rho \to 0$, that is, n-copies of the standard cylinder acted by A over each connected component of $\mathcal{L}$.

Recall that, on a neighborhood of X in $\overline{M}$, we have a rescaled metric $\overline{g} = r^2 g_M$ which extends to X and coincides with the metric g_X there. Now, an *admissible singular framing* $(\mathcal{F}, \kappa, \mathcal{L})$ over M consists of a union of disjoint simple curves $\mathcal{L}$ in M, a reference framing κ over $\mathcal{L}$, and a framing $\mathcal{F}$ over $M \setminus \mathcal{L}$, satisfying

i) the closure $\overline{\mathcal{L}}$ is smooth in $\overline{M}$, and $\overline{\mathcal{L}}$ is orthogonal to X in $\overline{g}$ at the intersection,
ii) the framings $r^{-1}\mathcal{F}$ and $r^{-1}\kappa$ extend smoothly to $\overline{M} \setminus \overline{\mathcal{L}}$ and $\overline{\mathcal{L}}$ respectively,
iii) the first vector e_1 of $\mathcal{F}$ is tangent to the gradient flow curves of r over $N_{(0,\epsilon)} \setminus \mathcal{L}$ for $0 < \epsilon < a$, and
iv) the framing $\mathcal{F}$ has an admissible singularity at $\mathcal{L}$.

Let $\ell_1, \ldots, \ell_g$ be closed curves in M representing the marked generators of $\pi_1(M)$, with the property that there exist discs $D_1, \ldots, D_{g-1}$ such that $M \setminus \cup D_i$ is the disjoint union of g solid tori $\ell_i \times D$, where D is the unit disc. Given an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$, define $\mathcal{L}^1$ to be the set of connected components of $\mathcal{L}$ that are closed, and define $\mathcal{L}^2 := \mathcal{L} \setminus \mathcal{L}^1$. Then $(\mathcal{F}, \kappa, \mathcal{L})$ will be called *standard* if

i) $\mathcal{F}$ has a special singularity of index 1 at each curve in $\mathcal{L}^1$ where the set $\mathcal{L}^1$ is a subset of $\{\ell_1, \ldots, \ell_g\}$ and
ii) the index of the admissible singularity of $\mathcal{F}$ at each curve in $\mathcal{L}^2$ is -1.

We define an admissible singular framing on a surface X with the metric g_X in a similar way. Let Z consist of finitely many points in X. A reference framing on Z is a choice of a frame (e_2, e_3) at each point $z \in Z$, orthonormal with respect to the metric g_X. A reference framing on Z defines a geodesic polar coordinate $\alpha : (0, \epsilon) \times Z \times S^1 \to \mathcal{N}^\epsilon(Z) \setminus Z$ which takes (ρ, z, v) to the point at distance ρ from $z \in Z$ along the geodesic with initial vector $\cos(v)e_2(y) + \sin(v)e_3(y)$. Parallel translation gives a corresponding reference framing

$\tilde{\alpha}$ over $(0,\epsilon)\times Z\times S^1$. The special singularity of index n at $z\in Z$ is the framing $\tilde{\alpha}\cdot A_n$ on $(0,\epsilon)\times\{z\}\times S^1$ where $\tilde{\alpha}$ denotes the reference framing and A_n is the matrix function given by

$$A_n(\rho,v)=\begin{pmatrix}\cos(nv) & -\sin(nv)\\ \sin(nv) & \cos(nv)\end{pmatrix}.$$

An admissible singularity of index n at z is the special singularity, right-multiplied by a matrix function $A(\rho,z,v)$ with the property that $\lim_{\rho\to 0}A(\rho,z,v)$ exists and is independent of v. An admissible singular framing $(\mathcal{F},\kappa,Z)$ on X consists of a finite set Z in X, a reference framing on Z, and a framing $\mathcal{F}$ of $X\setminus Z$ such that the limit of $\mathcal{F}$ as $\rho\to 0$ exists for all $v\in S^1$ and the map given by this limit is the same as the map given by the limit of an admissible singularity at each point of Z.

3.2. **Admissible singular framings associated to holomorphic 1-forms.** Suppose that X is a Riemann surface, with metric g_X compatible with its complex structure. We now describe how to assign, to a holomorphic 1-form Φ with only simple zeroes, an admissible singular framing with index -1 singular points at the zeroes of Φ.

The metric g_X is a collection $\{e^{\phi_\alpha}|dz_\alpha|^2\}_{\alpha\in A}$ on an atlas $\{(U_\alpha,z_\alpha)\}_{\alpha\in A}$ of X for which the functions $\phi_\alpha\in C^\infty(U_\alpha,\mathbb{R})$ satisfy

$$\phi_\alpha+\log|f'_{\alpha\beta}(z_\beta)|^2=\phi_\beta\qquad\text{on}\quad U_\alpha\cap U_\beta,\tag{3.2}$$

where $f_{\alpha\beta}=z_\alpha\circ z_\beta^{-1}:z_\beta(U_\alpha\cap U_\beta)\to z_\alpha(U_\alpha\cap U_\beta)$ are the holomorphic transition functions. A holomorphic 1-form Φ on X is a collection $\{h_\alpha dz_\alpha\}$ for the atlas $\{(U_\alpha,z_\alpha)\}$ for which h_α is a holomorphic function on U_α satisfying

$$h_\alpha f'_{\alpha\beta}(z_\beta)=h_\beta\qquad\text{on}\quad U_\alpha\cap U_\beta.\tag{3.3}$$

The phase function $e^{i\theta_\alpha}:=h_\alpha/|h_\alpha|$ is well defined over $X\setminus Z$ where Z denotes the zero set of Φ. The transformation law (3.3) implies

$$i\theta_\alpha+\log\frac{f'_{\alpha\beta}(z_\beta)}{|f'_{\alpha\beta}(z_\beta)|}=i\theta_\beta\qquad\text{on}\quad U_\alpha\cap U_\beta.\tag{3.4}$$

Note that θ_α is defined only up to an integer multiple of 2π. By (3.2), (3.4), it follows that $e^{\phi_\alpha/2+i\theta_\alpha}dz_\alpha$ defines an orthonormal co-framing ω_2,ω_3 given by

$$\omega_{2\alpha}=e^{\phi_\alpha/2}(\cos\theta_\alpha dx_\alpha-\sin\theta_\alpha dy_\alpha),\qquad \omega_{3\alpha}=e^{\phi_\alpha/2}(\sin\theta_\alpha dx_\alpha+\cos\theta_\alpha dy_\alpha)$$

on $U_\alpha\setminus Z$ where $z_\alpha=x_\alpha+iy_\alpha$. Now we obtain an orthonormal framing

$$\mathcal{F}_\Phi=(f_2,f_3)\qquad\text{where}\quad f_2=\omega_2^*,f_3=\omega_3^*$$

over $X\setminus Z$, which has admissible singularities at Z of index -1.

For the singular part Z, let $z_{i\alpha}$ denote the co-ordinate of a zero of Φ in a patch U_α. Then h_α has an expression $h_\alpha=(z_\alpha-z_{i\alpha})\tilde{h}_{i\alpha}$, where $\tilde{h}_{i\alpha}$ is non-vanishing at the zero. Now we put $e^{i\tilde{\theta}_{i,\alpha}}:=\tilde{h}_{i,\alpha}/|\tilde{h}_{i,\alpha}|$. Since $\tilde{h}_{i\alpha}$ is non-vanishing at the zero, $\tilde{\theta}_{i\alpha}$ is well-defined at the zero up to an integer multiple of 2π. By (3.2), (3.3), it follows that $e^{\frac{1}{2}(\phi_\alpha+i\tilde{\theta}_{i,\alpha})}dz_\alpha$ defines the following orthonormal co-framing at the zero,

$$\begin{aligned}\tilde{\omega}_{2\alpha}=&\ e^{\phi_\alpha/2}(\cos(\tilde{\theta}_\alpha/2)dx_\alpha-\sin(\tilde{\theta}_\alpha/2)dy_\alpha),\\ \tilde{\omega}_{3\alpha}=&\ e^{\phi_\alpha/2}(\sin(\tilde{\theta}_\alpha/2)dx_\alpha+\cos(\tilde{\theta}_\alpha/2)dy_\alpha),\end{aligned}\tag{3.5}$$

and the corresponding orthonormal framing $(\tilde{f}_2, \tilde{f}_3)$ at the zero. By the transformation law for $\tilde{h}$, this orthonormal framing transforms correctly under change of coordinate. Note however that this co-frame and frame are well defined only up to sign.

We select $g-1$ of the points in Z to have the framing $(\tilde{f}_2, \tilde{f}_3)$, and let the other $g-1$ points in Z have the framing $(\tilde{f}_2, -\tilde{f}_3)$; we denote the resulting framing at Z by κ_Φ. When we extend the framing $\mathcal{F}_\Phi$ to M, these will correspond to "outgoing" and "incoming" endpoints of curves in M respectively.

3.3. **Existence of admissible extensions.** On a subset of X, we can identify any $SO(2)$ framing with respect to g_X with an $SO(3)$ framing with respect to $\overline{g}$, by taking each framing (f_2, f_3) to the framing (f_1, f_2, f_3), where f_1 is the inward unit normal vector to X with respect to $\overline{g}$. We say that an admissible singular framing $(\mathcal{F}_X, \kappa_X, Z)$ has an *admissible extension* to M if there exists an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ over M such that $\partial\overline{\mathcal{L}} = Z$, and such that the extension of $r^{-1}\mathcal{F}$ and $r^{-1}\kappa$ equals the given framing $\mathcal{F}_X$ and κ_X, respectively, under the identification above.

Now, our goal is to show that, for a holomorphic 1-form Φ with only simple zeroes on X, the associated admissible singular framing $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ on X extends to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ on M. (A similar proof shows that any admissible singular framing on X extends to M.)

Before proving the existence of such an admissible extension, we establish two lemmas.

Lemma 3.1. *Suppose $\overline{W} = W \cup \partial W$ is a marked smooth 3-dimensional closed handlebody of genus p with metric $g_{\overline{W}}$, and suppose that $\mathcal{F}_{\partial W}$ is a smooth (non-singular) $SO(3)$ framing of ∂W. Then there exists an admissible extension of $\mathcal{F}_{\partial W}$ to W which has a special singularity of index 1 at $\mathcal{L}^1$. Its set of singular curves $\mathcal{L}^1$ may be taken to consist of at most p closed curves, each representing a distinct marked generator of $\pi_1(W)$.*

Proof. There exists a smooth embedding of W into $\mathbb{R}^3$, which gives a global framing $\mathcal{F}_0$ on W, by which we can identify any other framing on W with a map to $SO(3)$. Let $\mathcal{L}^0$ be the union of p closed simple curves representing the marked generators of $\pi_1(W)$. Given a connected curve ℓ in $\mathcal{L}^0$, there exists a disc D in W such that $W \setminus D$ is the disjoint union of a handlebody of genus $p-1$ and a solid torus T satisfying $T \cap \mathcal{L}^0 = \ell$ and $\partial T \simeq \ell \times S^1$. Since ∂D is homologically trivial in ∂W, it is a commutator in $\pi_1(\partial W)$ and so its image in $SO(3)$ under the framing $\mathcal{F}_{\partial W}$ is homotopically trivial. Hence $\mathcal{F}_{\partial W}$ can be smoothly extended to $D \subset \partial(W \setminus D)$. In this way the problem reduces to finding a framing on each solid torus T. If $\pi_1(T)$ is represented by ℓ, identify ∂T with $\ell \times S^1$. The image of this S^1 in $SO(3)$ given by $\mathcal{F}_{\partial W}$ is either homotopically trivial, in which case the framing extends smoothly to all of T, or it is homotopically nontrivial, in which case the framing has the same homotopy type as a special singularity framing of index 1 around ℓ and can thus be extended to a framing on $T \setminus \ell$ with this singularity. □

From now on, we put $a_1 = \frac{a}{4}$ for simplicity, where a is defined as in subsection 2.2.

Lemma 3.2. *Let $\overline{M} = M \cup X$ be a marked Schottky hyperbolic 3-manifold, and let $a > 0$ be such that the neighborhood $N_{[0,a]} \subset \overline{M}$ of X exists. Let Φ be a holomorphic 1-form with only simple zeroes on X and $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ be the associated admissible singular framing as defined above. Then $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ has an admissible extension to $N_{(0,a_1]}$.*

Proof. If Z is the singular set of the framing $\mathcal{F}_\Phi$ on X, then we can take the set of singular curves to be the g_M geodesics given by $\mathcal{L} = \{\phi_r(x) : x \in Z, r \in (0, a_1]\}$. Given an admissible singular framing $\mathcal{F}_\Phi = (f_1, f_2, f_3)$ over $X \setminus Z$ with respect to $\bar{g} = r^2 g_M$, one can find an admissible singular framing $\mathcal{F} = (e_1, e_2, e_3)$ with respect to g_M that is parallel near infinity and extends $\mathcal{F}_\Phi$, by rewriting the parallel transport equation for e_i with respect to g_M in terms of b_i, where $e_i(r) = rb_i(r) = r(b_i^1(r)\frac{\partial}{\partial t} + b_i^2(r)\frac{\partial}{\partial x} + b_i^3(r)\frac{\partial}{\partial y})$. The parallel transport equation along the gradient flow curve ϕ_r becomes

$$r\dot{b_i^m}(r) + b_i^m(r) + r\sum_{j,k} \Gamma^m_{j,k}(\phi_r)\dot{\phi}^j_r b_i^k(r) = 0,$$

and we use the solution, with initial conditions $b_i(0) = f_i$, to define e_i. We extend the reference framing on $\mathcal{L}$ in the same manner, using the reference framing on Z as the initial condition. □

Theorem 3.3. *If $\overline{M} = M \cup X$ is a marked Schottky hyperbolic 3-manifold and Φ is a holomorphic 1-form with only simple zeroes on X, then the associated admissible singular framing $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$ on X extends to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$ on M. The framing $(\mathcal{F}, \kappa, \mathcal{L})$ can be taken to be standard.*

Proof. We begin by defining the $\mathcal{L}^2$ part of the singular curve of $\mathcal{F}$. In Lemma 3.2, the $\mathcal{L}^2$ part in $N_{(0,\frac{a}{4}]}$ is defined to be the gradient flow curves. Now we extend them by taking pairs of two ends in X^{a_1} of those curves and making curves to connect them smoothly within $N_{(0,a)}$. We may assume that each connected curve ℓ_i, $i = 1, \dots, g-1$ in $\mathcal{L}^2$ meets level surface X^ϵ at two points for $a_1 \le \epsilon < \frac{a}{2}$ and at one point for $\epsilon = \frac{a}{2}$. By construction, the end points of $\mathcal{L}^2$ are given by the zero set $Z = \{p_1, \dots, p_{2g-2}\}$ of Φ. As we mentioned in the end of subsection 3.2, we may assume that if the reference framing is taken to be $(\tilde{f}_2, \tilde{f}_3)$ on one end of ℓ_i, then the reference framing is taken to be $(\tilde{f}_2, -\tilde{f}_3)$ on the other end of ℓ_i.

Let us choose a reference framing κ^2 on $\mathcal{L}^2$ which extends $(\tilde{f}_2, \tilde{f}_3)$ and $(\tilde{f}_2, -\tilde{f}_3)$ at each end point respectively, and which satisfies the parallel condition over $\mathcal{L}^2 \cap N_{(0,a_1]}$. We also let $\mathcal{F}$ be the admissible extension of $\mathcal{F}_\Phi$ on the set $N_{(0,\frac{a}{4}]}$ guaranteed to exist by Lemma 3.2. Note that $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{(0,a_1]}$ by definition.

Now we define $\mathcal{F}$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap N_{[a_1,a)}$ so that $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{[a_1,a)}$. Let β_i be a diffeomorphism from $\overline{\ell}_i \subset \overline{M}$ to $[-1,1]$ which maps the end with the reference framing $(\tilde{f}_2, \tilde{f}_3)$ to -1 and the end with the reference framing $(\tilde{f}_2, -\tilde{f}_3)$ to 1, and maps $\ell_i \cap N_{[a_1,a)}$ to $[-\frac{1}{2}, \frac{1}{2}]$. Let ξ be a smooth increasing function on the interval $[-1,1]$ whose derivative is supported in $(-\frac{1}{3}, \frac{1}{3})$ whose values are 0 on $[-1, -\frac{1}{3}]$ and π on $[\frac{1}{3}, 1]$. We define $\chi : \overline{\mathcal{L}^2} \to [0,\pi]$ by the composition of ξ and β_i over ℓ_i and let

$$A(\rho, v, y) = \begin{pmatrix} \cos\chi(y) & 0 & -\sin\chi(y) \\ 0 & 1 & 0 \\ \sin\chi(y) & 0 & \cos\chi(y) \end{pmatrix} \quad \text{on} \quad (0,\epsilon) \times (\mathcal{L}^2 \cap N_{[\frac{a}{3},a)}) \times S^1 \tag{3.6}$$

and A over $(0,\epsilon) \times (\mathcal{L}^2 \cap N_{[a_1,\frac{a}{3}]}) \times S^1$ is defined to connect the above matrix in (3.6) and the matrix A determining the admissible framing $\mathcal{F}$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap X^{a_1}$. We may assume that $\lim_{\rho\to 0} A(\rho, v, y)$ exists and is independent of v, for all $y \in \mathcal{L}^2$ and $v \in S^1$. Then, for the reference framing $\tilde{\alpha}$ of the deleted ϵ-tube around $\mathcal{L}$ obtained from κ^2, we define $\mathcal{F}$ by the equality $\mathcal{F} \circ \alpha = \tilde{\alpha} \cdot A_{-1} \cdot A$ over $\mathcal{N}^\epsilon(\mathcal{L}^2) \cap N_{[a_1,a)}$, which extends the previously constructed framing $\mathcal{F}$ over $N_{(0,a_1]}$. Note that this extension of $\mathcal{F}$ is independent of the choice of a reference framing κ^2 on $\mathcal{L}^2$ satisfying the conditions above. In particular, the extension of $\mathcal{F}$

does not depend on the choice of signs in κ_Φ. By definition, this framing $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2 \cap N_{[a_1,a)}$.

So far an admissible framing $\mathcal{F}$ has been constructed over $N_{(0,a_1]} \cup \mathcal{N}^\epsilon(\mathcal{L}^2)$. Now we extend it over $M \setminus (\mathcal{L}^1 \cup \mathcal{L}^2)$ by appropriately choosing $\mathcal{L}^1$. First let W_0 denote the closure of $M^{a_1} \setminus \mathcal{N}^\epsilon(\mathcal{L}^2)$ where $M^{a_1} = M \setminus N_{(0,a_1)}$. Then there is a homotopy which deforms W_0 to a closed handlebody W_1 of genus $2g-1$. Given a set of generators of $\pi_1(M) \simeq \pi_1(M^{a_1})$, there exist $(g-1)$-closed discs $D_i \subset W_1$, $i = 1, \ldots, g-1$ such that these decompose W_1 into one handlebody of genus g and solid tori T_i, $i = 1, \ldots, g-1$ satisfying the following conditions: the decomposed handlebody of genus g contains the homotopic images of loops realizing the given generators of $\pi_1(M^{\frac{a}{4}})$. For a generator $\tilde{\gamma}_i$ of $\pi_1(T_i)$, there is a closed curve γ_i in W_0 given by the (inverse) homotopic image of the loop realizing $\tilde{\gamma}_i$. By this construction, the set G of generators of $\pi_1(W_0)$ is given by the union of the chosen generators of $\pi_1(M^{a_1})$ by marking and the set of $\gamma_1, \ldots, \gamma_{g-1}$.

Applying Lemma 3.1 for the framing defined as above over the boundary of the closure of W_0, we obtain an admissible extension of $(\mathcal{F}_\Phi, \kappa_\Phi, Z)$. To show that we can take it to be standard, we have to modify the construction so that $\mathcal{L}^1$ consists of representatives of the marked generators of $\pi_1(M)$. Suppose that $\mathcal{L}^1$ contains a representative of a generator γ_i. Then we may replace the reference framing $\tilde{\alpha}$ with another framing with an additional rotation 2π along the corresponding part of $\mathcal{L}^2$. This will change the homotopy type of the admissible singular framing $\mathcal{F}$ along it since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. Hence it can be extended over the subset of W_0 corresponding to T_i without removing a curve representing γ_i. This means $\mathcal{L}^1$ can be taken to represent a subset of the given generators of $\pi_1(M)$. Then this completes the proof. □

4. Definition of the invariant $\mathbb{CS}$

4.1. **The form C on $PSL_2(\mathbb{C})$.** If H^3 is the hyperbolic space of dimension 3, the frame bundle $F(H^3)$ can be identified with $PSL_2(\mathbb{C})$ canonically. Let

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{h, e, f\}$ form a base of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of $PSL_2(\mathbb{C})$. Let $\{h^*_{\mathbb{C}}, e^*_{\mathbb{C}}, f^*_{\mathbb{C}}\}$ be its dual base of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$. In Section 3 in [18], Yoshida defines the form C as the left-invariant differential form on $PSL_2(\mathbb{C})$ whose value at the identity is given by $\frac{i}{\pi^2} h^*_{\mathbb{C}} \wedge e^*_{\mathbb{C}} \wedge f^*_{\mathbb{C}}$, and proves the following:

Proposition 4.1. *The form C on $PSL_2(\mathbb{C})$ is complex analytic, closed, and bi-invariant, and has the following expression*

$$\begin{aligned} C = \ & \frac{1}{4\pi^2}(4\,\theta_1 \wedge \theta_2 \wedge \theta_3 - d(\theta_1 \wedge \theta_{23} + \theta_2 \wedge \theta_{31} + \theta_3 \wedge \theta_{12})) \\ & + \frac{i}{4\pi^2}(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} - \theta_{12} \wedge \theta_1 \wedge \theta_2 - \theta_{13} \wedge \theta_1 \wedge \theta_3 - \theta_{23} \wedge \theta_2 \wedge \theta_3). \end{aligned}$$

Here θ_i and θ_{ij} denote the fundamental form and the connection form respectively on $PSL_2(\mathbb{C})$ of the Riemannian connection of H^3.

Since H^3 has constant sectional curvature -1, $\Omega_{ij} = -\theta_i \wedge \theta_j$ for $i, j = 1, 2, 3$. Thus C is a complex analytic form on $PSL_2(\mathbb{C})$ whose real part, up to scalar multiplication, is the volume form plus an exact form, and whose imaginary part, up to scalar multiplication, is the Chern-Simons form defined in [2]. Using the equalities $d\theta_i = -\sum_j \theta_{ij}\wedge\theta_j$, $d\theta_{ij} = -\sum_k \theta_{ik}\wedge\theta_{kj}+\Omega_{ij}$, one can obtain

Proposition 4.2. *The form C on $PSL_2(\mathbb{C})$ has the following expressions*

$$\begin{aligned} C &= -\frac{i}{4\pi^2}\,\eta\wedge d\eta \\ &= -\frac{1}{4\pi^2}(d\theta_{23}\wedge\theta_1 + d\theta_1\wedge\theta_{23}) + \frac{i}{4\pi^2}(d\theta_{23}\wedge\theta_{23} - d\theta_1\wedge\theta_1), \end{aligned}$$

where $\eta = \theta_1 - i\theta_{23}$.

For an oriented smooth hyperbolic manifold $M = \Gamma\backslash H^3$ of dimension 3, let $\tilde{M}$ be the universal cover of M and $d : \tilde{M} \to H^3$ be a developing map. Taking the differential of d, we obtain the $SO(3)$-bundle map $\tilde{d} : F(\tilde{M}) \to PSL_2(\mathbb{C})$. Since the form C is left invariant, $\tilde{d}^*C$ projects to a closed form on $F(M) = \Gamma\backslash F(\tilde{M})$ which by abuse of notation we denote also by C. Now, for the rest of this section, suppose that M is a marked Schottky hyperbolic 3-manifold. For an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$ over M, we introduce a map

$$s : (M\setminus\mathcal{L})\cup\mathcal{L} \to F(M) \tag{4.1}$$

defined by the admissible singular framing $\mathcal{F}$ over $M\setminus\mathcal{L}$ and the reference framing κ on $\mathcal{L}$. For $0<\epsilon<a_1$, we now define

$$\mathbb{CS}^\epsilon(M,s) = \int_{s(M^\epsilon\setminus\mathcal{L})} C - \sum_j \frac{n(j)}{2\pi}\int_{s(\ell_j^\epsilon)}(\theta_1 - i\theta_{23}) \tag{4.2}$$

where $M^\epsilon := M\setminus N_{(0,\epsilon)}$, ℓ_j denotes a connected component of $\mathcal{L}$, and $\ell_j^\epsilon := \ell_j\cap M^\epsilon$. Here the sum is over the connected components ℓ_j of $\mathcal{L}$ and $n(j)$ is the index of the admissible singularity of $\mathcal{F}$ at ℓ_j. The complex-valued invariant we define will be a suitably regularized value of $\mathbb{CS}^\epsilon(M,s)$ as $\epsilon\to 0$.

For a standard admissible framing $(\mathcal{F},\kappa,\mathcal{L})$ over M, the singular curve $\mathcal{L}$ consists of two parts: $\mathcal{L}^1$ is a union of simple closed curves and $\mathcal{L}^2$ is a union of curves connecting two end points in $X = \partial\overline{M}$. Then the quantity defined in (4.2) is given by

$$\mathbb{CS}^\epsilon(M,s) = \int_{s(M^\epsilon\setminus\mathcal{L})} C - \frac{1}{2\pi}\int_{s(\mathcal{L}^1)}(\theta_1 - i\theta_{23}) + \frac{1}{2\pi}\int_{s(\mathcal{L}^{2,\epsilon})}(\theta_1 - i\theta_{23}) \tag{4.3}$$

where $\mathcal{L}^{2,\epsilon} := \mathcal{L}^2\cap M^\epsilon$.

4.2. **Boundaries of $\overline{s(M^\epsilon\setminus\mathcal{L})}$.** For a standard admissible framing $(\mathcal{F},\kappa,\mathcal{L})$ over M, we investigate the structure of the boundaries of $\overline{s(M^\epsilon\setminus\mathcal{L})}$ where the closure is taken in $F(M)$. The boundary $\partial(\overline{s(M^\epsilon\setminus\mathcal{L})})$ consists of three parts which we are going to describe below.

One part of the boundary $\partial(\overline{s(M^\epsilon\setminus\mathcal{L})})$ is given by the closure of $s(X^\epsilon\setminus\mathcal{L}^2)$ in $F(M)$, which we denote by $B^{0,\epsilon}$. Note that the boundary of $B^{0,\epsilon}$ consists of a disjoint union of circles.

The second part of the boundary $\partial(\overline{s(M^\epsilon\setminus\mathcal{L})})$ is given by $\bigcup_{y\in\mathcal{L}^1}\lim_{\delta\to 0} s(S_\delta(y))$, where $S_\delta(y)$ denotes the circle consisting of points in the orthogonal disc to $\mathcal{L}^1$ of distance δ from $y\in\mathcal{L}^1$. For $y\in\mathcal{L}^1$, the limit of $s(S_\delta(y))$ as $\delta\to 0$ exists since the framing $\mathcal{F}$ has a special singularity of index 1 at $\mathcal{L}^1$. We denote this part of boundary, which does not depend on ϵ, by B^1. Actually B^1 is given by the standard cylinder over $\mathcal{L}^1$: there is a map

$$\psi : \mathcal{L}^1\times S^1 \to p^{-1}(\mathcal{L}^1)\subset F(M)$$

which takes the point $(y, v) \in \mathcal{L}^1 \times S^1$ to the framing

$$\psi(y, v) := (e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)) \tag{4.4}$$

at the point $y \in \mathcal{L}^1$. Here (e_1, e_2, e_3) is the reference framing κ^1 on $\mathcal{L}^1$. The boundary orientation of B^1 is induced from $\mathcal{F}$ and is given by $(\psi_* \frac{\partial}{\partial y}, \psi_* \frac{\partial}{\partial v})$ so that ψ is orientation-preserving.

The remaining part of boundary $\partial(\overline{s(M^\epsilon \setminus \mathcal{L})})$ is given by $\bigcup_{y \in \mathcal{L}^{2,\epsilon}} \lim_{\delta \to 0} s(S_\delta(y))$. For $y \in \mathcal{L}^2$, the limit of $s(S_\delta(y))$ as $\delta \to 0$ exists since the framing $\mathcal{F}$ has an admissible singularity of index -1 at $\mathcal{L}^2$. We denote this part by $B^{2,\epsilon}$. Note that $B^{2,\epsilon}$ has circle boundaries which are the boundaries of $B^{0,\epsilon}$ with the opposite orientation. As the case of B^1, $B^2 = \lim_{\epsilon \to 0} B^{2,\epsilon}$ can be described in terms of the standard cylinder over $\mathcal{L}^2$ with some modification. There is a map

$$\psi : \mathcal{L}^2 \times S^1 \to p^{-1}(\mathcal{L}^2) \subset F(M)$$

which takes the point $(y, v) \in \mathcal{L}^2 \times S^1$ to the framing given by

$$\psi(y, v) = (e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y)), \tag{4.5}$$

where (e_1, e_2, e_3) is the reference framing κ^2 on $\mathcal{L}^2$. We denote by $\tilde{B}^2$ the image of ψ. We take the orientation of $\tilde{B}^2$ to be given by $(\psi_* \frac{\partial}{\partial y}, -\psi_* \frac{\partial}{\partial v})$, so that ψ is orientation-reversing by definition. The $\tilde{B}^2$ and B^2 do not coincide completely, but we can describe their difference explicitly:

Lemma 4.3. *The fiberwise right multiplication of A appearing in equation* (3.1) *induces an orientation preserving diffeomorphism $\mathcal{A}$ of $p^{-1}(\overline{\mathcal{N}^\epsilon}(\mathcal{L}^2)) \subset F(M)$ mapping $\tilde{B}^2$ to B^2 over $\mathcal{L}^2$.*

Proof. The claim follows directly from the definition of admissible singularity. □

4.3. **Real part of $\mathbb{CS}^\epsilon(M, s)$.** We start with

Lemma 4.4. *For s corresponding to an admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$, the following equalities hold over $N_{(0,a_1)} \setminus \mathcal{L}^2$,*

$$\omega_{12} = II(e_2, e_2)\omega_2 + II(e_3, e_2)\omega_3, \qquad \omega_{13} = II(e_2, e_3)\omega_2 + II(e_3, e_3)\omega_3,$$

where $\omega_i = s^*\theta_i$, $\omega_{ij} = s^*\theta_{ij}$ denote the fundamental forms and connection forms pulled back by s respectively, and $II(*, *)$ denotes the second fundamental form.

Proof. By definition of $\mathcal{F} = (e_1, e_2, e_3)$, e_1 is tangent to a geodesic which is also trajectory of the gradient flow of the defining function r and e_2, e_3 are tangent to the level surface X^ϵ with $r = \epsilon$. We use the equality $\omega_{ij}(e_k) = -g_M(\nabla_{e_k} e_i, e_j)$ to obtain $\omega_{1j}(e_1) = 0$ and $\omega_{1j}(e_k) = -g_M(\nabla_{e_k} e_1, e_j) = II(e_k, e_j)$ for $j = 2, 3$, $k = 2, 3$. This completes the proof. □

The mean curvature H is defined to be the trace of II. (Note that H is defined to the half of the trace of II in some of the literature.) In [13], W-volume of M^ϵ is defined by

$$W(M^\epsilon) := \mathrm{Vol}(M^\epsilon) - \frac{1}{4} \int_{X^\epsilon} H \mathrm{dvol}$$

where $\mathrm{Vol}(M^\epsilon)$ denotes the volume of M^ϵ and dvol denotes the area form over X^ϵ induced by g_M. One nice property of W-volume proved in Lemma 4.5 in [13] is the following equality: for $0 < \epsilon < a$,

$$W(M^\epsilon) = 2\pi(1 - g) \log \epsilon + W_{\text{f.p.}}(M^\epsilon), \tag{4.6}$$

where $W(M) := \lim_{\epsilon\to 0} W_{\text{f.p.}}(M^\epsilon)$ exists and defines the renormalized volume $W(M)$ of M as in Section 8 of [13].

Proposition 4.5. *For s defined by a standard admissible singular framing $(\mathcal{F}, \kappa, \mathcal{L})$,*

$$\operatorname{Re}\mathbb{CS}^\epsilon(M,s) = \frac{1}{\pi^2} W(M^\epsilon) \qquad \textit{for} \quad 0<\epsilon<a_1.$$

Proof. By the definition, we have

$$\begin{aligned}\int_{\overline{s(M^\epsilon\setminus\mathcal{L})}} \operatorname{Re} C =& \frac{1}{4\pi^2}\int_{\overline{s(M^\epsilon\setminus\mathcal{L})}} \big(4\theta_1\wedge\theta_2\wedge\theta_3 - d(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})\big)\\ =&\frac{1}{\pi^2}\operatorname{Vol}(M_\epsilon) - \frac{1}{4\pi^2}\int_{\partial(\overline{s(M^\epsilon\setminus\mathcal{L})})} \theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}.\end{aligned} \tag{4.7}$$

For the second equality in (4.7), we apply Stokes' theorem. Now we consider the integrals over the boundary $\partial(\overline{s(M^\epsilon\setminus\mathcal{L})}) = B^{0,\epsilon}\cup B^1\cup B^{2,\epsilon}$. For the boundary integral over $B^{0,\epsilon}$, we have

$$\begin{aligned}-\frac{1}{4\pi^2}\int_{B^{0,\epsilon}}\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12} =& \ \frac{1}{4\pi^2}\int_{X^\epsilon}\omega_1\wedge\omega_{23}+\omega_2\wedge\omega_{31}+\omega_3\wedge\omega_{12}\\ =-\frac{1}{4\pi^2}\int_{X^\epsilon}\operatorname{tr}II\ \omega_2\wedge\omega_3 =& -\frac{1}{4\pi^2}\int_{X^\epsilon} H\text{dvol},\end{aligned}$$

where X^ϵ is oriented by $\omega_2\wedge\omega_3$ and the second equality follows from Lemma 4.4.

For the boundary integral over B^1, recall that the boundary B^1 is diffeomorphic to $\mathcal{L}^1\times S^1$ by ψ in (4.4), and that $\psi_*\frac{\partial}{\partial v}$ is a vertical vector field and $\psi^*\theta_{1j}(\frac{\partial}{\partial v}) = 0$ for $j=2,3$ by definition of B^1, hence $\psi^*(\theta_2\wedge\theta_{31})(\frac{\partial}{\partial v},*)=0$, $\psi^*(\theta_3\wedge\theta_{12})(\frac{\partial}{\partial v},*)=0$. Moreover, by definition, $\psi^*\theta_{23}(\frac{\partial}{\partial v}) = -1$. This implies

$$\begin{aligned}-\frac{1}{4\pi^2}\int_{B^1}\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12} =& -\frac{1}{4\pi^2}\int_{\mathcal{L}^1\times S^1}\psi^*(\theta_1\wedge\theta_{23})\\ =& \ \frac{1}{2\pi}\int_{\mathcal{L}^1}\psi^*\theta_1 = \frac{1}{2\pi}\int_{\mathcal{L}^1} s^*\theta_1 = \frac{1}{2\pi}\int_{s(\mathcal{L}^1)}\theta_1.\end{aligned}$$

Hence the boundary integral over B^1 cancels the real part of the second integral in (4.3).

For the boundary integral $B^{2,\epsilon}$,

$$\begin{aligned}&-\frac{1}{4\pi^2}\int_{B^{2,\epsilon}}\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}\\ =&-\frac{1}{4\pi^2}\int_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}),\end{aligned}$$

where ψ is given by (4.5). Using $\mathcal{A}^*\theta = A^{-1}\cdot\theta$ and $\mathcal{A}^*\Theta = A^{-1}\cdot dA + A^{-1}\cdot\Theta\cdot A$ with $\theta = (\theta_1,\theta_2,\theta_3)^t$, $\Theta = (\theta_{ij})$,

$$\begin{aligned}&\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})\\ &=\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12}+\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3 + a_{j2}A_3\cdot dA_1 + a_{j3}A_1\cdot dA_2)\end{aligned} \tag{4.8}$$

where a_{jk} denotes the entry in A and A_j denotes the column vector of A, and $A_j\cdot dA_k$ denotes the inner product of two vectors. By Lemma 4.3 and (4.8), and repeating the computation of the integral over B^1,

$$
\begin{aligned}
&-\frac{1}{4\pi^2}\int_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\mathcal{A}^*(\theta_1\wedge\theta_{23}+\theta_2\wedge\theta_{31}+\theta_3\wedge\theta_{12})\\
&=-\frac{1}{2\pi}\int_{s(\mathcal{L}^{2,\epsilon})}\theta_1-\frac{1}{4\pi^2}\int_{\mathcal{L}^{2,\epsilon}\times S^1}\psi^*\big(\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3+a_{j2}A_3\cdot dA_1+a_{j3}A_1\cdot dA_2)\big)\\
&=-\frac{1}{2\pi}\int_{s(\mathcal{L}^{2,\epsilon})}\theta_1.
\end{aligned}
$$

Here we use that $\psi:\mathcal{L}^2\times S^1\to\tilde{B}^2$ in (4.5) is orientation reversing, and that the form involving A vanishes on the vertical vector field $\psi_*\frac{\partial}{\partial v}$. Hence the boundary integral over $B^{2,\epsilon}$ cancels the real part of the third integral in (4.3). This completes the proof. $\square$

4.4. **Imaginary part of $\mathbb{CS}^\epsilon(M,s)$.** Now we prove

Proposition 4.6. *For s corresponding to an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$, the imaginary part of $\mathbb{CS}^\epsilon(M,s)$ converges to a finite value as $\epsilon\to 0$.*

Proof. Over $N_{(0,a_1)}\setminus\mathcal{L}^2$, the pull back of the imaginary part of C by s is given by

$$
\frac{1}{4\pi^2}\big(\omega_{12}\wedge\omega_{13}\wedge\omega_{23}-\omega_{12}\wedge\omega_1\wedge\omega_2-\omega_{13}\wedge\omega_1\wedge\omega_3-\omega_{23}\wedge\omega_2\wedge\omega_3\big). \tag{4.9}
$$

The first and the last terms in (4.9) vanish respectively since they are sum of triple wedge products of ω_2,ω_3 by Lemma 4.4. The second and the third terms in (4.9) cancel each other by Lemma 4.4 and the fact $II(e_2,e_3)=II(e_3,e_2)$. Hence the imaginary part of the first integral in (4.2) is finite and independent of $0<\epsilon<a_1$. For the imaginary part of the line integral over $\mathcal{L}$, note that for $\ell_j\in\mathcal{L}^2$, the integral $\int_{\ell_j\cap N_{[\epsilon,a_1]}}\omega_{23}$ measures the total rotation of κ with respect to parallel translation on $\ell_j\cap N_{[\epsilon,a_1]}$. Since $r^{-1}\kappa$ extends smoothly to $\overline{M}$ by definition, the limit of the line integral as $\epsilon\to 0$ has a finite value. This completes the proof. $\square$

Proposition 4.7. *For a given marked Schottky hyperbolic 3-manifold M, if s_0, s_1 are defined by standard admissible framings $(\mathcal{F}_0,\kappa_0,\mathcal{L}_0)$ and $(\mathcal{F}_1,\kappa_1,\mathcal{L}_1)$ on M which are related by a homotopy of standard admissible framings which are fixed outside of M^{a_1}, then*

$$
\operatorname{Im}\mathbb{CS}^\epsilon(M,s_0)=\operatorname{Im}\mathbb{CS}^\epsilon(M,s_1).
$$

Proof. Let $(\mathcal{F}_u,\kappa_u,\mathcal{L}_u)$, with $u\in[0,1]$ be the homotopy connecting $(\mathcal{F}_0,\kappa_0,\mathcal{L}_0)$ and $(\mathcal{F}_1,\kappa_1,\mathcal{L}_1)$. The framing $\mathcal{F}_u$ defines a section $s:W_\epsilon\to F(M)$ over $W_\epsilon:=[0,1]\times M^\epsilon\setminus\{(u,y_u)\mid y_u\in\mathcal{L}_u,u\in[0,1]\}$. Denoting by Q the imaginary part of C, we have

$$
0=\int_{s(W_\epsilon)}dQ=\int_{s_1(M^\epsilon\setminus\mathcal{L}_1)}Q-\int_{s_0(M^\epsilon\setminus\mathcal{L}_0)}Q+\int_{B_W}Q. \tag{4.10}
$$

The boundary B_W consists of three parts $\hat{B}^0$, $\hat{B}^1$, and $\hat{B}^2$, consisting of the trajectories under the homotopy $\mathcal{F}_u$ of $B^{0,\epsilon}$, B^1, and $B^{2,\epsilon}$ respectively. For the integral over the part $\hat{B}^0$, $\theta_i(s_*\frac{\partial}{\partial u})=0$ and $\theta_{ij}(s_*\frac{\partial}{\partial u})=0$ over $\hat{B}^0=B^{0,\epsilon}\subset F(M)$. Therefore

$$
\int_{\hat{B}^0}Q=0. \tag{4.11}
$$

The boundary $\hat{B}^1$ is diffeomorphic to $[0,1] \times \mathcal{L}^1 \times S^1$ by

$$\psi(u, y, v) = \{u\} \times (e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y))$$

where $(e_1(y), e_2(y), e_3(y))$ denotes the reference framing $\kappa_u(y)$ for $y \in \mathcal{L}^1$. Here and below, we identify $\mathcal{L}^1_u$ with $\mathcal{L}^1 = \mathcal{L}^1_0$ implicitly. The orientation $(\frac{\partial}{\partial u}, \frac{\partial}{\partial y}, -\frac{\partial}{\partial v})$ on $[0,1] \times \mathcal{L}^1 \times S^1$ makes ψ orientation preserving. As before, $\psi^*\Omega_{ij}(\frac{\partial}{\partial v}, *) = 0$ for $1 \le i,j \le 3$, $(\psi^*\theta_{12})(\frac{\partial}{\partial v}) = 0$, $(\psi^*\theta_{13})(\frac{\partial}{\partial v}) = 0$, and $(\psi^*\theta_{23})(\frac{\partial}{\partial v}) = -1$. From above facts, we have

$$\psi^*Q = \frac{1}{4\pi^2}\psi^*(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{23} \wedge \Omega_{23}) = \frac{1}{4\pi^2}\psi^*(\theta_{23} \wedge d\theta_{23}),$$

and

$$\psi^*\theta_{23} = -dv + q^*s^*\theta_{23},$$

where $q : [0,1] \times \mathcal{L}^1 \times S^1 \to [0,1] \times \mathcal{L}^1$ is the natural projection, $s_u : \mathcal{L}^1 \to F(M)$ is the section defined by κ_u, and $s : [0,1] \times \mathcal{L}^1 \to F(M)$ is the corresponding family given by $s(u, \cdot) = s_u$. It follows that $\psi^*Q = -\frac{1}{4\pi^2} dv \wedge d(q^*s^*\theta_{23})$. With the above orientation convention, by Stokes' theorem, we have

$$\begin{aligned}\int_{\hat{B}^1} Q =& \int_{[0,1]\times\mathcal{L}^1\times S^1} \psi^*Q = -\frac{1}{4\pi^2}\int_{[0,1]\times\mathcal{L}^1\times S^1} dv \wedge d(q^*s^*\theta_{23}) \\ =& \frac{1}{2\pi}\int_{[0,1]\times\mathcal{L}^1} d(s^*\theta_{23}) = \frac{1}{2\pi}\Big(\int_{\mathcal{L}^1} s_1^*\theta_{23} - \int_{\mathcal{L}^1} s_0^*\theta_{23}\Big).\end{aligned} \tag{4.12}$$

The right hand side of (4.12) is the same as the difference of the imaginary parts of the second integrals for $u = 1$ and $u = 0$ in the definition of $\mathbb{CS}^\epsilon(M, s)$ in (4.3).

For the boundary integral over $\hat{B}^{2,\epsilon}$, as in the proof of Proposition 4.5 we have

$$\int_{\hat{B}^{2,\epsilon}} Q = \int_{[0,1]\times\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\mathcal{A}^*Q$$

where ψ is the orientation reversing diffeomorphism defined in (4.5). We also have

$$\begin{aligned}\mathcal{A}^*Q =\ & Q + \frac{1}{24\pi^2}\mathrm{Tr}((A^{-1}dA)^3) \\ & + \frac{1}{4\pi^2} d\,(\theta_{12} \wedge d\hat{A}_1 \cdot \hat{A}_2 + \theta_{13} \wedge d\hat{A}_1 \cdot \hat{A}_3 + \theta_{23} \wedge d\hat{A}_2 \cdot \hat{A}_3)\end{aligned} \tag{4.13}$$

where $\hat{A}_j$ denotes the row vector of A. Hence, in a similar way as (4.12),

$$\begin{aligned}&\int_{\hat{B}^{2,\epsilon}} Q \\ =& -\frac{1}{2\pi}\Big(\int_{\mathcal{L}^{2,\epsilon}} s_1^*\theta_{23} - \int_{\mathcal{L}^{2,\epsilon}} s_0^*\theta_{23}\Big) - \frac{1}{2\pi}\Big(\int_{\mathcal{L}^{2,\epsilon}} \psi_1^* d\hat{A}_2 \cdot \hat{A}_3 - \int_{\mathcal{L}^{2,\epsilon}} \psi_0^* d\hat{A}_2 \cdot \hat{A}_3\Big) \\ =& -\frac{1}{2\pi}\Big(\int_{\mathcal{L}^{2,\epsilon}} s_1^*\theta_{23} - \int_{\mathcal{L}^{2,\epsilon}} s_0^*\theta_{23}\Big)\end{aligned} \tag{4.14}$$

where ψ_1 and ψ_0 represent ψ taken at $u = 1$ and $u = 0$. Here the first equality follows from that $\psi^*\theta_{12}$, $\psi^*\theta_{13}$, and the form involving A vanish on the vertical vector field $\psi_* \frac{\partial}{\partial v}$. The expression $(1/2\pi)\int_{\mathcal{L}^{2,\epsilon}} \psi_u^* d\hat{A}_2 \cdot \hat{A}_3$ can be shown to be the total rotation angle of $\hat{A}_2$ about the axis $\hat{A}_1$ along $\mathcal{L}^{2,\epsilon}$. Since A is fixed at the endpoints of $\mathcal{L}^{2,\epsilon}$ through the homotopy, this total rotation angle does not change, so the second equality follows. The right hand side of (4.14) is the same as the difference of the imaginary parts of the third integrals for $u = 1$ and $u = 0$

in the definition of $\mathbb{CS}^\epsilon(M,s)$ in (4.3). Combining (4.10), (4.11), (4.12) and (4.14) completes the proof. $\square$

4.5. **Definition of the invariant $\mathbb{CS}(M,s)$ and the function $\mathbb{CS}$.** For $s: M \to F(M)$ corresponding to an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$ as explained after equation (4.1), we define the Chern-Simons invariant of (M,s) to be

$$CS(M,s) := \frac{1}{2}\lim_{\epsilon\to 0} \operatorname{Im} \mathbb{CS}^\epsilon(M,s),$$

where the limit exists by Proposition 4.6. By Proposition 4.7, $CS(M,s)$ is independent of a homotopic change of an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$ inside of M^{a_1}. We can now define the invariant $\mathbb{CS}(M,s)$.

Definition 4.8. For $s : M \to F(M)$ corresponding to an admissible singular framing $(\mathcal{F},\kappa,\mathcal{L})$,

$$\mathbb{CS}(M,s) := \lim_{\epsilon\to 0}(\mathbb{CS}^\epsilon(M,s) + \frac{2}{\pi}(g-1)\log\epsilon).$$

By (4.6) and Proposition 4.5, as we stated in (1.2), we have

$$\mathbb{CS}(M,s) = \frac{1}{\pi^2}W(M) + 2iCS(M,s).$$

Now, suppose we are given a compact marked Riemann surface X and a holomorphic 1-form Φ on X, with corresponding admissible singular framing $(\mathcal{F}_\Phi,\kappa_\Phi,Z)$ over X. Then we have associated to this data a unique marked Schottky hyperbolic 3-manifold M_X and a standard admissible extension $(\mathcal{F},\kappa,\mathcal{L})$ over M_X corresponding to $s_\Phi : M_X \to F(M_X)$. We now consider to what extent the invariant $\mathbb{CS}(M_X,s_\Phi)$ depends on our choice of admissible extension s_Φ. We have already shown in Proposition 4.7 that it is independent of a homotopic change of $(\mathcal{F},\kappa,\mathcal{L})$ in $M_X^{a_1}$. Now we show

Proposition 4.9. *The quantity* $\exp(4\pi\mathbb{CS}(M_X,s_\Phi))$ *is independent of the choice of signs in* κ_Φ *and of the choice of* κ*.*

Proof. Note that the modulus of $\exp(4\pi\mathbb{CS}(M_X,s_\Phi))$ depends only on M_X by Proposition 4.5. For the argument of $\exp(4\pi\mathbb{CS}(M_X,s_\Phi))$, there is a choice of a reference framing κ which can rotate along $\mathcal{L}$, but a change of a rotation number results in only an integer difference in the imaginary part of $\mathbb{CS}(M_X,s_\Phi)$ through the second and third integrals in (4.3). There are sign ambiguities in the definition of the reference framing at zeroes of Φ mentioned just after (3.5). Hence the imaginary part of $\mathbb{CS}(M_X,s_\Phi)$ is well-defined only up to addition of a half-integer, but this ambiguity will disappear for $\exp(4\pi\mathbb{CS}(M_X,s_\Phi))$. $\square$

To state the main result of this section, we need to introduce an auxiliary space. For each point (X,Φ) in $\tilde{\mathcal{H}}_g(1,\dots,1)$, we attach the data of a choice of isotopy class of $g-1$ simple, pairwise disjoint curves in M_X whose endpoints are the zeroes of Φ. The resulting space $\tilde{\mathcal{H}}^*_g(1,\dots,1)$ is locally isomorphic to $\tilde{\mathcal{H}}_g(1,\dots,1)$, and there is a natural projection map to $\tilde{\mathcal{H}}_g(1,\dots,1)$ corresponding to forgetting the added data. Note that each connected component of $\tilde{\mathcal{H}}^*_g(1,\dots,1)$ covers $\tilde{\mathcal{H}}_g(1,\dots,1)$ by this projection map.

Theorem 4.10. *The expression* $\exp(4\pi\mathbb{CS}(M_X,s_\Phi))$ *determines a globally well-defined function* $\exp(4\pi\mathbb{CS}) : \tilde{\mathcal{H}}^*_g(1,\dots,1) \to \mathbb{C}$.

Proof. Given a point in $\tilde{\mathcal{H}}_g^*(1,\dots,1)$, we use Theorem 3.3 to construct a standard singular admissible framing on M_X, whose $\mathcal{L}^2$ curves are isotopic to the given $g-1$ curves. It is clear from the construction that any two such framings are related by a homotopy, which is an isotopy of the corresponding set of curves $\mathcal{L}$. It then follows from Propositions 4.7 and 4.9 that the value of $\exp(4\pi\mathbb{CS}(M_X, s_\Phi))$ is uniquely determined by this data. □

Remark 4.11. The proof of the main theorem in Section 8 will show that, in fact, the function $\exp(4\pi\mathbb{CS})$, restricted to any connected component of $\tilde{\mathcal{H}}_g^*(1,\dots,1)$, descends to a well-defined function on $\tilde{\mathcal{H}}_g(1,\dots,1)$. Restricting to a different connected component of $\tilde{\mathcal{H}}_g^*(1,\dots,1)$ will give a function on $\tilde{\mathcal{H}}_g(1,\dots,1)$ differing from the original by a multiplicative constant.

5. Variation of the invariant $\mathbb{CS}$

Suppose we are given a contractible open set U in $\tilde{\mathcal{H}}_g^*(1,\dots,1)$. By the results of the previous section, the invariant $\mathbb{CS}(M_X, s_\Phi)$ determines a function $\mathbb{CS}: U \to \mathbb{C}$, which is well-defined up to addition of $\frac{1}{2}ni$ for $n \in \mathbb{Z}$. In this section we find expressions for the derivatives $\partial\mathbb{CS}$ and $\bar{\partial}\mathbb{CS}$ of this function.

5.1. **Basic notations for variation.** Each point $u \in U$ determines a compact marked Riemann surface X_u together with a holomorphic 1-form Φ_u on X_u. We fix a basepoint $u_0 \in U$, and for simplicity we write $X = X_{u_0}$ and similarly below. We will always assume that X_u is uniformized by a marked Schottky group, $X_u = \Gamma_u\backslash\Omega_u$, where Γ_u is a marked Schottky group with marked normalized generators $\{L_1(u),\dots,L_g(u)\}$ and ordinary set Ω_u. The group Γ_u simultaneously defines a marked Schottky hyperbolic 3-manifold $M_u := M_{X_u} = \Gamma_u\backslash H^3$. For each $u \in U$, we have a quasi-conformal mapping $f_u : \Omega \to \Omega_u$. Define $P_u^\epsilon : \Omega_u \to H^3$ to be the map translating points along the integral curve ϕ_t of $\nabla_{\bar{g}_u} r_u$ emanating from $z \in \Omega_u$ for ϵ units of time, where $\bar{g}$ and r are defined as in subsection 2.2. Then we define a map $\mathbf{f}_u : \cup_{0<\epsilon<a_1} P^\epsilon(\Omega) \to H^3$ by

$$\mathbf{f}_u|_{P^\epsilon(\Omega)} = P_u^\epsilon \circ \mathbf{f}_u \circ (P^\epsilon)^{-1}. \tag{5.1}$$

(Here $a_1 = \frac{a}{4}$, where a is defined as in subsection 2.2.) This map extends to a diffeomorphism $\mathbf{f}_u : H^3 \to H^3$, satisfying $\mathbf{f}_u \circ \gamma = \gamma_u \circ \mathbf{f}_u$ for all $\gamma_u \in \Gamma_u$.

Corresponding to the family Φ_u and the given homotopy class of $g-1$ curves in M_u connecting the zeroes of Φ_u, we take a smooth family of sections $s_u := s_{\Phi_u} : (M_u \setminus \mathcal{L}_u) \cup \mathcal{L}_u \to F(M_u)$, constructed as in Theorem 4.10. Here $\mathcal{L}^2$ is taken to be isotopic to the given $g-1$ curves, and $\mathcal{L}_u = \mathbf{f}_u(\mathcal{L})$. We also denote by $\mathcal{L}_u$ and s_u the corresponding liftings $\mathcal{L}_u \subset H^3$ and $s_u : (H^3 \setminus \mathcal{L}_u) \cup \mathcal{L}_u \to F(H^3) \cong PSL_2(\mathbb{C})$. The family defines a map $s : U \times H^3 = \{(u,x) \mid u \in U, x \in (H^3 \setminus \mathcal{L}_u) \cup \mathcal{L}_u\} \to PSL_2(\mathbb{C})$. We let K be the unique map $K : U \times s_0((H^3 \setminus \mathcal{L}) \cup \mathcal{L}) \to PSL_2(\mathbb{C})$ satisfying

$$K \circ (\mathrm{id}, s_0) = s \circ (\mathrm{id}, \mathbf{f}),$$

where $s \circ (\mathrm{id}, \mathbf{f})(u,x) := s_u \circ \mathbf{f}_u(x)$ for $(u,x) \in U \times H^3$. As observed in subsection 4.2, the closure $\overline{s_0(H^3 \setminus \mathcal{L})}$ of $s_0(H^3 \setminus \mathcal{L})$ in $PSL_2(\mathbb{C})$ provides a natural compactification of $s_0(H^3 \setminus \mathcal{L})$, and K extends smoothly to $U \times (\overline{s_0(H^3 \setminus \mathcal{L})} \cup s_0(\mathcal{L}))$ (we also denote the extension by K). Note that the generators $L_r(u)$ of Γ_u, $r = 1,\dots,g$, can be considered as giving holomorphic functions

$$L_r : U \to PSL_2(\mathbb{C}).$$

We let $\mathbf{D}$ be a fundamental domain for the action of Γ on H^3, such that $\partial\mathbf{D} \subset H^3$ consists of $2g$ smooth surfaces D_r, $-L_r(0)D_r$, for $r = 1, \ldots, g$ (the negative sign indicates opposite orientation). Define $\mathbf{D}_u := \mathbf{f}_u(\mathbf{D})$. Considering H^3 as $\{(t, x, y) \in \mathbb{R}^3 | t > 0\}$, define D and C_r to be the intersection of the closure of $\mathbf{D}$ and D_r respectively with the set $t = 0$. Then D is a fundamental domain of the action of Γ on Ω, ∂D consists of smooth curves C_r, $-L_r(0)C_r$, and we define $D_u := f_u(D)$. We denote $\mathbf{D}' := \mathbf{D} \setminus \mathcal{L}$, and define $\Delta := s_0(\mathbf{D}')$. As above, the closure $\overline{\Delta}$ of Δ in $PSL_2(\mathbb{C})$ provides a natural compactification of Δ. Let $T_r := \overline{s_0(D_r)}$ for $r = 1, \ldots, g$. The boundary components of $\overline{\Delta}$ consist of $B^0 \cup B^1 \cup B^2$ as defined in the subsection 4.2, and $\cup_{r=1}^g (T_r - L_r(0)T_r)$. We denote by $\mathbf{D}^\epsilon$ and $\overline{\Delta}^\epsilon$ the subsets of $\mathbf{D}$ and $\overline{\Delta}$ respectively corresponding to M^ϵ. Define $D^\epsilon := \mathbf{D} \cap P^\epsilon(D)$. The boundary components of $\overline{\Delta}^\epsilon$ consist of $B^{0,\epsilon} \cup B^1 \cup B^{2,\epsilon}$ and $\cup_{r=1}^g (T_r^\epsilon - L_r(0)T_r^\epsilon)$ where $B^{0,\epsilon}$ is diffeomorphic to B^0, and $B^{2,\epsilon}$ and $\cup_{r=1}^g (T_r^\epsilon - L_r(0)T_r^\epsilon)$ are subsets of B^2 and $\cup_{r=1}^g (T_r - L_r(0)T_r)$ respectively. The notations $\mathbf{D}'_u$, Δ_u, etc. denote the corresponding constructions for $\mathbf{D}_u$.

Since we will always be working in a fixed fundamental domain $\mathbf{D}_u$, from now on, we will write $\mathcal{L}_u = \mathcal{L}_u^1 \cup \mathcal{L}_u^2$ to mean the intersection $\mathcal{L}_u \cap \mathbf{D}_u$. The boundary points of $\mathcal{L}_u^1 \cup \mathcal{L}_u^2$ consist of finitely many matched pairs $y_j(u)$ and $L_{r(j)}(u)y_j(u)$, $j \in \mathcal{J}$, together with $2g - 2$ points which are the zeros of the holomorphic 1-form Φ_u. We may assume that every curve in $\mathcal{L}_u^1$ has exactly two points $y_j(u)$, $L_{r(j)}(u)y_j(u)$ in its boundary, and we assume the orientation of $\mathcal{L}_u^1$ given by the reference framing κ_u^1 is such that the component connecting $L_{r(j)}(u)y_j(u)$ to $y_j(u)$ is oriented towards $y_j(u)$.

Under the canonical map from $\tilde{\mathcal{H}}_g^*(1, \ldots, 1)$ to $\mathfrak{S}_g$, a holomorphic tangent vector ϖ in $T^{1,0}U$ at u_0 maps to a holomorphic tangent vector in $T^{1,0}\mathfrak{S}_g$, which corresponds to a harmonic Beltrami differential $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$. Then μ defines a quasi-conformal mapping $f_{w\mu}$: $X \to X_w$ for all w in some neighborhood W of the origin in $\mathbb{C}$. There exists a holomorphic family $\{\Phi(w)\}$, where $\Phi(w)$ is a holomorphic 1-form on X_w, such that the derivative at u_0 of the complex curve in U given by the family $\{(X_w, \Phi(w))\}$ is ϖ. (Here we are using the local isomorphism of $\tilde{\mathcal{H}}_g^*(1, \ldots, 1)$ and $\tilde{\mathcal{H}}_g(1, \ldots, 1)$.) In this way we obtain a complex curve $u : W \to U$, such that $\frac{\partial u}{\partial w} = \varpi$ and $\frac{\partial u}{\partial \bar{w}} = 0$ (with w a local coordinate in W).

For the curve $u : W \to U$ we define $f : W \times \Omega \to \mathbb{C}$ by $f(w, z) = f_{u(w)}(z) = f_{w\mu}(z)$ and $\mathbf{f} := W \times H^3 \to H^3$ by $\mathbf{f}(w, x) = \mathbf{f}_{u(w)}(x)$. We also define

$$H = K \circ (u, \mathrm{id}) : W \times (\overline{s_0(H^3 \setminus \mathcal{L})} \cup s_0(\mathcal{L})) \to PSL_2(\mathbb{C}),$$

and

$$\sigma = s \circ (u, \mathbf{f}) : W \times H^3 \to PSL_2(\mathbb{C}).$$

5.2. **Contributions of boundaries.** For technical reasons we consider the holomorphic variation of $\overline{\mathbb{CS}}$ rather than $\mathbb{CS}$.

To derive a variation formula for $\overline{\mathbb{CS}}$, we start with the following equality:

$$\begin{aligned}
0 = \int'_{\overline{\Delta}^\epsilon} H^* d\overline{C} = \int'_{\overline{\Delta}^\epsilon} (d_W + d_\Delta) H^* \overline{C} = d_W \int'_{\overline{\Delta}^\epsilon} H^* \overline{C} - \int'_{\partial \overline{\Delta}^\epsilon} H^* \overline{C} \\
= d_W \int'_{\overline{\Delta}^\epsilon} H^* \overline{C} - \int'_{B^{0,\epsilon} \cup B^1 \cup B^{2,\epsilon}} H^* \overline{C} - \sum_{r=1}^g \int'_{T_r^\epsilon - L_r T_r^\epsilon} H^* \overline{C}.
\end{aligned} \tag{5.2}$$

Here the notation $\int'_{\overline{\Delta}^\epsilon}$ denotes the *partial integral*: we consider the integrand as a form on $\overline{\Delta}$ taking values in forms on W, and integrate over $\{w\} \times \overline{\Delta}^\epsilon$, obtaining a 1-form on W. The notation $d = d_W + d_\Delta$ denotes the splitting of d on $W \times \Delta$ in the obvious way. Note that we use the orientation from $W \times \Delta$; for this reason, we have $d_W \int'_{\Delta^\epsilon} H^*\overline{C} = \int'_{\Delta^\epsilon} d_W H^*\overline{C}$, but when we apply Stokes' theorem, we have $\int'_{\Delta^\epsilon} d_\Delta H^*\overline{C} = -\int'_{\partial\Delta^\epsilon} H^*\overline{C}$. We use a similar convention for partial integrals throughout this section.

The next three lemmas deal with the partial integrals $\int'_{B^1} H^*\overline{C}$, $\int'_{B^{2,\epsilon}} H^*\overline{C}$, and $\int'_{B^{0,\epsilon}} H^*\overline{C}$ respectively.

Lemma 5.1. *Let $u : W \to U$ be a complex curve as defined above, with $w \in W$. Then we have the following equality of 1-forms over W:*

$$\int'_{B^1} H^*\overline{C} - \frac{1}{2\pi} d_W \int'_{\mathcal{L}^1} \sigma^*(\theta_1 + i\theta_{23}) = -\frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^1} \sigma^*(\theta_1 + i\theta_{23})|_y.$$

Proof. Recall that the integral over B^1 is independent of ϵ for small $\epsilon > 0$. As in the proof of Proposition 4.6, we have the diffeomorphism

$$\psi : W \times \mathcal{L}^1 \times S^1 \longrightarrow W \times B^1$$

defined by

$$\psi(w, y, v) = \{w\} \times H(w, \cdot)^{-1}(e_1(y), \cos(v)e_2(y) + \sin(v)e_3(y), -\sin(v)e_2(y) + \cos(v)e_3(y))$$

for $w \in W$, $y \in \mathcal{L}^1$ and $v \in S^1$. The notation $H(w, \cdot)^{-1}$ denotes the inverse of $H(w, \cdot)$ restricted to its image. The orientation of $\mathcal{L}^1 \times S^1$ is given by $(\frac{\partial}{\partial y}, \frac{\partial}{\partial v})$. As in the proof of Propositions 4.5 and 4.6, we have $(\psi^* H^*\theta_i)(\frac{\partial}{\partial v}) = (\psi^* H^*\theta_{1i})(\frac{\partial}{\partial v}) = 0$ $(i = 1, 2, 3)$ and $(\psi^* H^*\theta_{23})(\frac{\partial}{\partial v}) = -1$. It follows that

$$\psi^* H^*\overline{C} = -\frac{1}{4\pi^2}\psi^* H^*(d(\theta_1 \wedge \theta_{23})) - \frac{i}{4\pi^2}\psi^* H^*(\theta_{23} \wedge d\theta_{23}).$$

Let $q : W \times \mathcal{L}^1 \times S^1 \to W \times \mathcal{L}^1$ be the natural projection. Then

$$\psi^* H^*\theta_{23} = -dv + q^*\sigma^*\theta_{23}.$$

It follows that $\psi^* H^*(\theta_{23} \wedge d\theta_{23}) = -dv \wedge d(q^*\sigma^*\theta_{23})$. From the above orientation convention, by Stokes' theorem, we have

$$\begin{aligned}
&\frac{1}{4\pi^2}\int'_{\mathcal{L}^1 \times S^1} \psi^* H^*(d(\theta_1 \wedge \theta_{23})) \\
=&\frac{1}{4\pi^2} d_W \int'_{\mathcal{L}^1 \times S^1} \psi^* H^*(\theta_1 \wedge \theta_{23}) - \frac{1}{4\pi^2}\int'_{\partial\mathcal{L}^1 \times S^1} \psi^* H^*(\theta_1 \wedge \theta_{23}) \\
=&-\frac{1}{2\pi} d_W \int'_{\mathcal{L}^1} \sigma^*\theta_1 + \frac{1}{2\pi}\sum_{y \in \partial\mathcal{L}^1} \sigma^*\theta_1|_y.
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\frac{1}{4\pi^2}\int'_{\mathcal{L}^1 \times S^1} \psi^* H^*(\theta_{23} \wedge d\theta_{23}) \\
=&-\frac{1}{4\pi^2}\int'_{\mathcal{L}^1 \times S^1} dv \wedge d(q^*\sigma^*\theta_{23}) = -\frac{1}{2\pi}\int'_{\mathcal{L}^1} d(\sigma^*\theta_{23}) \\
=&-\frac{1}{2\pi} d_W \int'_{\mathcal{L}^1} \sigma^*\theta_{23} + \frac{1}{2\pi}\sum_{y \in \partial\mathcal{L}^1} \sigma^*\theta_{23}|_y.
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) proves the lemma. □

Lemma 5.2. *We have the following equality of 1-forms over W:*

$$\lim_{\epsilon\to 0}\Big(\int'_{B^{2,\epsilon}} H^*\overline{C} + \frac{1}{2\pi} d_W \int'_{\mathcal{L}^{2,\epsilon}} \sigma^*(\theta_1 + i\theta_{23})\Big) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^{2,\epsilon}} \sigma^*(\theta_1+i\theta_{23})|_y.$$

Proof. We define the map $\tilde{H} : W\times\tilde{B}^2 \to \tilde{B}^2_u$ by $\tilde{H} = \mathcal{A}^{-1}\circ H\circ(\mathrm{id},\mathcal{A})$. As in the proof of Propositions 4.5 and 4.6, by (4.8) and (4.13) and denoting $\mathcal{A}^{-1}(B^{2,\epsilon})$ by $\tilde{B}^{2,\epsilon}$,

$$\begin{aligned}
\int'_{B^{2,\epsilon}} H^*\overline{C} =& \int'_{\tilde{B}^{2,\epsilon}} (\mathrm{id},\mathcal{A})^* H^*\overline{C} = \int'_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\tilde{H}^*\mathcal{A}^*\overline{C}\\
=& -\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\tilde{H}^*(d(\theta_1\wedge\theta_{23})) + i\psi^*\tilde{H}^*(\theta_{23}\wedge d\theta_{23})\\
&-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\tilde{H}^* d\Big(\sum_{j=1}^{3}\theta_j\wedge(a_{j1}A_2\cdot dA_3 + a_{j2}A_3\cdot dA_1 + a_{j3}A_1\cdot dA_2)\Big)\\
&-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} i\psi^*\tilde{H}^* d(\theta_{12}\wedge d\hat{A}_1\cdot\hat{A}_2 + \theta_{13}\wedge d\hat{A}_1\cdot\hat{A}_3 + \theta_{23}\wedge d\hat{A}_2\cdot\hat{A}_3)\\
&-\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} \frac{i}{6}\psi^*\tilde{H}^*\mathrm{Tr}((A^{-1}dA)^3).
\end{aligned}$$

By $\psi^*\tilde{H}^*(\theta_i)(\frac{\partial}{\partial v}) = 0$, $\psi^*\tilde{H}^*(\theta_{ij})(\frac{\partial}{\partial v}) = 0$, $\psi^*\tilde{H}^*(dA_j)(\frac{\partial}{\partial v}) = 0$, and $\psi^*\tilde{H}^*(d\hat{A}_k)(\frac{\partial}{\partial v}) = 0$, all the integrals vanish except the integral of $\psi^*\tilde{H}^* d(\theta_{23}\wedge d\hat{A}_2\cdot\hat{A}_3)$ for the terms on the last three lines of the above equalities. But we have the following equality:

$$\begin{aligned}
&\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\tilde{H}^* d(\theta_{23}\wedge d\hat{A}_2\cdot\hat{A}_3)\\
=&\frac{1}{2\pi} d_W\int'_{\mathcal{L}^{2,\epsilon}} \psi^*\tilde{H}^*\big(d\hat{A}_2\cdot\hat{A}_3\big) - \frac{1}{2\pi}\sum_{y\in\partial\mathcal{L}^{2,\epsilon}} \psi^*\tilde{H}^*\big(d\hat{A}_2\cdot\hat{A}_3\big)|_y.
\end{aligned}$$

The first term in the second line can be shown to give the variation with respect to w of the sum of the total rotation angles of $\hat{A}_2$ around $\hat{A}_1$ along the components of $\mathcal{L}^{2,\epsilon}$. But by our assumptions on the framing, A limits to the identity at the boundary $\partial\mathcal{L}^2\cap D$. Hence the limit of this term as $\epsilon\to 0$ gives an integer, which is invariant under the deformation. The last term is 0 since the contributions from boundary points in the interior of M cancel by an invariance under identification by the $L_r(u(w))$, and at the remaining boundary points, $\psi^*\tilde{H}^*(d\hat{A}_2\cdot\hat{A}_3)|_y \to 0$ as $\epsilon\to 0$. From these equalities, we have

$$\lim_{\epsilon\to 0}\int'_{B^{2,\epsilon}} H^*\overline{C} = \lim_{\epsilon\to 0}\ -\frac{1}{4\pi^2}\int'_{\mathcal{L}^{2,\epsilon}\times S^1} \psi^*\tilde{H}^*(d(\theta_1\wedge\theta_{23})) + i\psi^*\tilde{H}^*(\theta_{23}\wedge d\theta_{23}).$$

Now, repeating the derivation in (5.3) and (5.4) and recalling that ψ is an orientation reversing diffeomorphism in this case, completes the proof. □

Now we deal with the partial integral over $B^{0,\epsilon}$. Recall that D^ϵ is the subset in $\mathbf{D} = \mathbf{D}_0$ corresponding to X^ϵ. First we have

Lemma 5.3. *We have the following equality of 1-forms on W:*

$$\begin{aligned}\int'_{B^{0,\epsilon}} H^*\overline{C} &= \int'_{D^\epsilon\setminus\mathcal{L}} \sigma^*\overline{C}\\ &= \frac{1}{4\pi^2}\int'_{(D^\epsilon\setminus\mathcal{L})} \big(\, d_D\omega_{23}\wedge(\chi_1+i\chi_{23}) + \omega_{23}\wedge\big(d_D(\chi_1+i\chi_{23}) + id_W\omega_{23}\big)\,\big)\end{aligned}$$

where $d = d_W + d_D$ over $W\times D^\epsilon$. Here χ_1 and χ_{23} are defined by $\sigma^*\theta_1 = \omega_1+\chi_1$ and $\sigma^*\theta_{23} = \omega_{23}+\chi_{23}$, where $\omega_1|_{TW} = \omega_{23}|_{TW} = 0$ and $\chi_1|_{TD^\epsilon} = \chi_{23}|_{TD^\epsilon} = 0$.

Note that ω_1, ω_{23}, χ_1 and χ_{23} depend on $w\in W$.

Proof. By Proposition 4.2, it is easy to see

$$\begin{aligned}\sigma^*\overline{C} = &-\frac{1}{4\pi^2}\big(\, d(\omega_{23}+\chi_{23})\wedge(\omega_1+i\omega_{23}+\chi_1+i\chi_{23})\,\big)\\ &+\frac{i}{4\pi^2}\big(\, d(\omega_1+\chi_1)\wedge(\omega_1+i\omega_{23}+\chi_1+i\chi_{23})\,\big).\end{aligned}$$

Now, note that $d_D\omega_1 = 0$ by Lemma 4.4, and that ω_1 vanishes on tangent vectors of $(D^\epsilon\setminus\mathcal{L})$. Also note that

$$0 = d_W\int'_{D^\epsilon\setminus\mathcal{L}}\omega_{23}\wedge\omega_1 = \int'_{D^\epsilon\setminus\mathcal{L}}(d_W\omega_{23})\wedge\omega_1 - \int'_{D^\epsilon\setminus\mathcal{L}}\omega_{23}\wedge(d_W\omega_1),$$

so $\int'_{D^\epsilon\setminus\mathcal{L}}\omega_{23}\wedge(d_W\omega_1) = 0$. Now, recalling that the orientation of D^ϵ is opposite to that of $B^{0,\epsilon}$, the result follows from direct computation. $\square$

5.3. **Limit of contribution over $B^{0,\epsilon}$.** Now we want to push the expression of Lemma 5.3 down to the boundary $D\subset\hat{\mathbb{C}}\subset\partial H^3$. This will be accomplished in Proposition 5.7. First we need to prove some preliminary results, which will also be useful later.

By the uniformization of X by Γ, we identify X with $\Gamma\backslash\Omega$. Then the hyperbolic metric g_X of constant curvature -1 on X (or the flat metric g_X of area 1 in the case that X has genus 1) gives a metric $e^{\phi(z)}|dz|^2$ on Ω, invariant under the action of Γ. The invariance implies that

$$\phi(z) = \phi(\gamma z) + \log|\gamma'(z)|^2 \tag{5.5}$$

for all $z\in\Omega$ and $\gamma\in\Gamma$.

Proposition 5.4. *The set D^ϵ in H^3 is given by $D^\epsilon = \{\,(t,x,y)\in H^3\,|\,t = \mathfrak{t}(\epsilon,x,y)\,\}$, where $\mathfrak{t}$ is a function satisfying*

$$\mathfrak{t}(\epsilon,x,y) = \epsilon e^{-\frac{\phi(x,y)}{2}} + k(\epsilon,x,y)\epsilon^3,$$

where k, k_x and k_y exist and are bounded on $\mathbf{D}\cup D$.

Proof. Let us recall that there is a unique defining function r over a collar neighborhood N of X in $\overline{M}$ such that the rescaled metric $\bar{g} := r^2 g_M$ extends smoothly to $\overline{M}$, its restriction to X is the hyperbolic metric g_X and $|dr|^2_{\bar{g}} = 1$. Let us denote the lifted defining function over the inverse image of N in H^3 by the same notation r, and write $\hat{r} := \frac{r}{t}$. Then the three conditions on $\bar{g}$ imply that $\hat{r}$ extends smoothly to $\mathbf{D}\cup D$, $\lim_{t\to 0}\hat{r}(t,x,y) = e^{\frac{\phi}{2}}$, and $\hat{r}_t = -\frac{1}{2}(\hat{r}_t^2+\hat{r}_x^2+\hat{r}_y^2)\hat{r}^{-1}t$ respectively. Since $|\hat{r}_t|\le Ct$ for a uniform constant C, we have

$|(\hat{r}-e^{\frac{\phi}{2}})_t| \leq Ct$. Since $\hat{r}$ is smooth on $\mathbf{D}\cup D$, this means $|\hat{r}-e^{\frac{\phi}{2}}| \leq Ct^2$ for a uniform constant C, and therefore

$$\hat{r}(t,x,y) = e^{\frac{\phi(x,y)}{2}} + \alpha(t,x,y)t^2, \tag{5.6}$$

where α is uniformly bounded.

Similarly, since $\hat{r}_t = 0$ on D, we have $\hat{r}_{xt} = 0$ on D. Again, since $\hat{r}$ is smooth, we obtain $|(\hat{r} - e^{\frac{\phi}{2}})_x| \leq Ct^2$ for some uniform constant C. This implies

$$\hat{r}_x(t,x,y) = (e^{\frac{\phi(x,y)}{2}})_x + \alpha_x(t,x,y)t^2$$

where α_x is uniformly bounded, and similarly for $\hat{r}_y$. This implies the claimed expression for $\mathfrak{t}$. Now, since $k = -\alpha e^{-\frac{\phi}{2}}\hat{r}^{-3}$, and $\hat{r}$ is nowhere zero on D, the result follows. □

A holomorphic 1-form Φ with only simple zeroes over X is given by $h(z)dz$ over Ω with $h(\gamma z)\gamma'(z) = h(z)$ for $\gamma \in \Gamma$. The phase function $e^{i\theta(z)} := h(z)/|h(z)|$ is well defined over $\Omega \setminus \cup_{\gamma\in\Gamma}\gamma Z$ where $Z := \{z_1, \ldots, z_{2g-2}\}$ denotes the zero set of $h(z)$ in a fixed fundamental domain D of Γ. The transformation law of $h(z)$ implies

$$i\theta(z) = i\theta(\gamma z) + \log\frac{\gamma'(z)}{|\gamma'(z)|} \tag{5.7}$$

for $\gamma \in \Gamma$. Note that θ is defined only up to an integer multiple of 2π. By (5.5), (5.7), it follows that $e^{\phi(z)/2+i\theta(z)}dz = \omega_2 + i\omega_3$ is invariant under the action of Γ; in particular,

$$\omega_2 = e^{\phi/2}(\cos\theta dx - \sin\theta dy), \qquad \omega_3 = e^{\phi/2}(\sin\theta dx + \cos\theta dy)$$

provides us with an orthonormal invariant co-frame (ω_2, ω_3) over $\Omega\setminus\cup_{\gamma\in\Gamma}\gamma Z$. Now we obtain an orthonormal framing

$$\mathcal{F}_\Phi = (f_2, f_3) \qquad \text{where} \quad f_2 = \omega_2^*, f_3 = \omega_3^*$$

over $D' := D \setminus Z$.

Near a zero $z_k \in Z$, $h(z)$ has an expression $h(z) = (z - z_k)\tilde{h}_k(z)$ such that $\tilde{h}_k(z)$ is non-vanishing at z_k. Now we put $e^{i\tilde{\theta}_k(z)} := \tilde{h}_k(z)/|\tilde{h}_k(z)|$. Since $\tilde{h}_k(z)$ is non-vanishing at $z = z_k$, $\tilde{\theta}_k(z_k)$ is well-defined only up to an integer multiple of 2π. As in (3.5), we define

$$\begin{aligned}\tilde{\omega}_2 &= e^{\phi/2}(\cos(\tilde{\theta}_k/2)dx - \sin(\tilde{\theta}_k/2)dy),\\ \tilde{\omega}_3 &= e^{\phi/2}(\sin(\tilde{\theta}_k/2)dx + \cos(\tilde{\theta}_k/2)dy)\end{aligned} \tag{5.8}$$

at $z_k \in Z$. Then the duals $(\tilde{f}_2, \tilde{f}_3)$ of $(\tilde{\omega}_2, \tilde{\omega}_3)$ define an orthonormal framing at $z_k \in Z$. That this orthonormal framing is well-defined up to sign follows from the fact that $h(\gamma z)\gamma'(z) = h(z)$ and the from the following equality for $\gamma \in \Gamma$ and $z, z_k \in \Omega$:

$$(\gamma z - \gamma z_k) = (z - z_k)\gamma_z(z)^{\frac{1}{2}}\gamma_z(z_k)^{\frac{1}{2}}.$$

Proposition 5.5. *The one form ω_{23} on $\mathbf{D}'$ extends smoothly to a form on $\mathbf{D}' \cup D'$. We have*

$$\lim_{t\to 0}\omega_{23} = \frac{i}{2}\big(\,(\phi - 2i\theta)_z dz - (\phi + 2i\theta)_{\bar{z}}d\bar{z}\,\big),$$

where the convergence in the global coordinate on H^3 is uniform on $\mathbf{D}' \cup D'$.

Note that the extension of ω_{23} to D' coincides with the connection form of the hyperbolic metric $e^\phi|dz|^2$, with respect to our choice of orthonormal frame $\mathcal{F}_\Phi$.

Proof. By the Koszul formula, we have

$$\omega_{23} = g([e_2, e_3], e_2)\omega_2 + g([e_2, e_3], e_3)\omega_3$$

for an orthonormal frame (e_1, e_2, e_3) where e_1 is orthogonal to TD^ϵ. By the asymptotics of the boundary defining function r in (5.6), we have

$$e_1 = t(1+\tfrac{1}{4}t^2(\phi_x^2 + \phi_y^2))^{-\frac{1}{2}} (\tfrac{1}{2}t\phi_x\partial_x + \tfrac{1}{2}t\phi_y\partial_y + \partial_t) + O(t^3),$$
$$e_2 = \alpha_{22}\bar{e}_2 + \alpha_{23}\bar{e}_3, \quad e_3 = \alpha_{32}\bar{e}_2 + \alpha_{33}\bar{e}_3$$

with

$$\bar{e}_2 = t(1 + \tfrac{1}{4}t^2\phi_x^2)^{-\frac{1}{2}} (\partial_x - \tfrac{1}{2}t\phi_x\partial_t) + O(t^3), \qquad \bar{e}_2 = t(1 + \tfrac{1}{4}t^2\phi_y^2)^{-\frac{1}{2}} (\partial_y - \tfrac{1}{2}t\phi_y\partial_t) + O(t^3).$$

Here and below, we use $O(t^k)$ to indicate a function of the form $a(t, x, y)t^k$ with respect to the global coordinate on H^3, where a is uniformly bounded in $\mathbf{D}' \cup D'$. To compute $g([e_2, e_3], e_2)$, $g([e_2, e_3], e_3)$, we consider $[e_2, e_3]$ first. By an elementary computation,

$$\begin{aligned} [e_2, e_3] = \ & (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})[\bar{e}_2, \bar{e}_3] \\ & + \big(\alpha_{22}\bar{e}_2(\alpha_{32}) - \alpha_{32}\bar{e}_2(\alpha_{22}) + \alpha_{23}\bar{e}_3(\alpha_{32}) - \alpha_{33}\bar{e}_3(\alpha_{22})\big)\bar{e}_2 \\ & + \big(\alpha_{22}\bar{e}_2(\alpha_{33}) - \alpha_{32}\bar{e}_2(\alpha_{23}) + \alpha_{23}\bar{e}_3(\alpha_{33}) - \alpha_{33}\bar{e}_3(\alpha_{23})\big)\bar{e}_3. \end{aligned} \tag{5.9}$$

Using Proposition 5.4, we have

$$[\bar{e}_2, \bar{e}_3] = (\tfrac{1}{2}t^2\phi_y)\partial_x - (\tfrac{1}{2}t^2\phi_x)\partial_y + O(t^3),$$

from which we also have

$$\begin{aligned} g([\bar{e}_2, \bar{e}_3], e_2) =& \alpha_{22}(\tfrac{1}{2}t\phi_y) + \alpha_{23}(-\tfrac{1}{2}t\phi_x) + O(t^2) = \tfrac{1}{2}t(\cos\theta\,\phi_y + \sin\theta\,\phi_x) + O(t^2), \\ g([\bar{e}_2, \bar{e}_3], e_3) =& \alpha_{32}(\tfrac{1}{2}t\phi_y) + \alpha_{33}(-\tfrac{1}{2}t\phi_x) + O(t^2) = \tfrac{1}{2}t(\sin\theta\,\phi_y - \cos\theta\,\phi_x) + O(t^2). \end{aligned}$$

Here we used the fact $\alpha_{22} = \alpha_{33} = \cos\theta + O(t)$, $\alpha_{23} = -\alpha_{32} = -\sin\theta + O(t)$. Denoting by E the sum of the terms in the second and third lines on the right hand side of (5.9),

$$\begin{aligned} g(E, e_2) =& - t(\cos\theta\,\theta_x - \sin\theta\,\theta_y) + O(t^2), \\ g(E, e_3) =& - t(\sin\theta\,\theta_x + \cos\theta\,\theta_y) + O(t^2). \end{aligned}$$

Finally we need

$$\omega_2 = t^{-1}(\cos\theta dx - \sin\theta dy + O(t)), \qquad \omega_3 = t^{-1}(\sin\theta dx + \cos\theta dy + O(t)).$$

Combining all the proved equalities, we have

$$\begin{aligned} \omega_{23} = \ & (\frac{1}{2}\cos\theta\,\phi_y + \frac{1}{2}\sin\theta\,\phi_x - \cos\theta\,\theta_x + \sin\theta\,\theta_y)(\cos\theta\,dx - \sin\theta\,dy) \\ & + (\frac{1}{2}\sin\theta\,\phi_y - \frac{1}{2}\cos\theta\,\phi_x - \sin\theta\,\theta_x - \cos\theta\,\theta_y)(\sin\theta\,dx + \cos\theta\,dy) + O(t) \\ = \ & d\theta + \frac{1}{2}(\phi_y dx - \phi_x dy) + O(t). \end{aligned}$$

This completes the proof. □

Now, we define $c_1 = \chi_1(\frac{\partial}{\partial w})$ and $c_{23} = \chi_{23}(\frac{\partial}{\partial w})$, where χ_1 and χ_{23} were defined in Lemma 5.3, and w is a local coordinate in W. We will write $'$ for the derivative with respect to w, for instance, $\phi' = \frac{\partial}{\partial w}\phi$.

Proposition 5.6. *The functions c_1, c_{23} on $W \times \mathbf{D}'$ extend smoothly to functions on $W \times (\mathbf{D}' \cup D')$. We have*

$$\lim_{t\to 0} c_1 = -\frac{1}{2}\phi' \circ f, \qquad \lim_{t\to 0} c_{23} = \theta' \circ f + i(\frac{\phi}{2} - i\theta)_z f',$$

and the convergence in the global coordinate on H^3 is uniform on $\mathbf{D}' \cup D'$. We also have $\lim_{t\to 0} \chi_1(\frac{\partial}{\partial \bar w}) = \bar c_1$, $\lim_{t\to 0} \chi_{23}(\frac{\partial}{\partial \bar w}) = \bar c_{23}$.

Proof. Observe that c_1 is given by

$$(s \circ (u, \mathbf{f}))^* \theta_1(\tfrac{\partial}{\partial w}) = \theta_1(s_* u_* \tfrac{\partial}{\partial w}) + s^* \theta_1(\mathbf{f}_* u_* \tfrac{\partial}{\partial w}) = \omega_1(\mathbf{f}'),$$

where the second equality holds since $s_* u_* \frac{\partial}{\partial w}$ is vertical. Recall that the level surface D^ϵ is given by $\{\, (t,x,y) \in H^3 \,|\, t = \mathfrak{t}(\epsilon, x, y) = \epsilon e^{-\frac{\phi(x,y)}{2}} + O(\epsilon^3) \,\}$, and that the definition of $\mathbf{f}$ near the boundary given by (5.1) involves translation along gradient curves for r. Since translation from D to D^ϵ introduces an error of $O(\epsilon^2)$, and since f_z and f'_z are bounded on D, we have

$$\mathbf{f}(w, (t,z)) = \big(\mathfrak{t}\big(r(t,z), f(z)\big), f(z)\big) + O(t^2).$$

Here and below we understand $O(t^2)$ to be uniform as discussed in the previous proposition. Therefore we have

$$\mathbf{f}' = f' \tfrac{\partial}{\partial z} - \big(\tfrac{1}{2} t (\phi' \circ f + \phi_z f')\big) \tfrac{\partial}{\partial t} + O(t^2).$$

The one form ω_1 is the dual of the first component e_1 of the orthonormal frame over the level surface D^ϵ so that

$$\omega_1 = (1 + \tfrac{1}{4} t^2 (\phi_x^2 + \phi_y^2))^{-1/2} t^{-1} \big(\tfrac{1}{2} t \phi_z dz + \tfrac{1}{2} t \phi_{\bar z} d\bar z + dt\big) + O(t^3).$$

Hence, we have

$$\omega_1(\mathbf{f}') = \frac{1}{2}\phi_z f' - \frac{1}{2}\phi' \circ f - \frac{1}{2}\phi_z f' + O(t) = -\frac{1}{2}\phi' \circ f + O(t),$$

from which it follows

$$\lim_{t\to 0} c_1 = \lim_{t\to 0} \omega_1(\mathbf{f}') = -\frac{1}{2}\phi' \circ f. \tag{5.10}$$

As above, c_{23} is given by

$$(s \circ (u, \mathbf{f}))^* \theta_{23}(\tfrac{\partial}{\partial w}) = \theta_{23}(s_* u_* \tfrac{\partial}{\partial w}) + s^* \theta_{23}(\mathbf{f}_* u_* \tfrac{\partial}{\partial w}) = \theta_{23}(s') + \omega_{23}(\mathbf{f}').$$

Now, we have $\theta_{23} = \mathcal{L}^*_{g^{-1}}(-2(ih)^*)$ and $\lim_{t\to 0} (\mathcal{L}_g)_* s_* u_* (\frac{\partial}{\partial w}) = -\frac{1}{2}\theta_w(ih)$, where $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$ and $\mathcal{L}_g$ is the left translation by $g \in PSL_2(\mathbb{C})$ (see Section 3 of [18]). Hence,

$$\lim_{t\to 0} c_{23} = \lim_{t\to 0} \big(\theta_{23}(s') + \omega_{23}(\mathbf{f}')\big) = \theta' \circ f + i(\frac{\phi}{2} - i\theta)_z f'. \tag{5.11}$$

The equalities (5.10) and (5.11) complete the proof of the first two equalities. Replacing $\frac{\partial}{\partial w}$ with $\frac{\partial}{\partial \bar w}$ in the computations above gives the last part of the statement. $\square$

We denote by the same notations ω_{23}, c_1, c_{23}, the restriction to $W \times D'$ of the extensions of ω_{23}, c_1, c_{23} respectively, obtained in Propositions 5.5 and 5.6.

Now let us introduce some additional notation. The local coordinate expression for the members of the family $\{\Phi(w)\}$ can be identified with a map $h : \{(w, \Omega_{u(w)}) : w \in W\} \to \mathbb{C}$. We define $z_k : W \to \mathbb{C}$ to be the coordinates in $\Omega_{u(w)}$ of the zeroes of $\Phi(w)$, that is, $h(w, z_k(w)) = 0$ for all $w \in W$. Near each $z_k(w)$, we define $\tilde{h}_k$ by

$$h(w, f(w,z)) = \big(f(w,z) - f(w, z_k(w))\big)\, \tilde{h}_k(w, f(w,z)) \tag{5.12}$$

for all $w \in W$.

Proposition 5.7. *The limit of the 1-form*

$$\lim_{\epsilon \to 0} {\int_{B^{0,\epsilon}}}' H^* \overline{C} \tag{5.13}$$

over W is finite, and its $(1,0)$ part equals

$$\frac{1}{4\pi^2} {\int_{D'}}' \big(d_D\omega_{23} \wedge (c_1 + ic_{23})dw + \omega_{23} \wedge (d_D(c_1 + ic_{23}) \wedge dw + i\partial_w \omega_{23}) \big) \tag{5.14}$$

where $d = d_W + d_D = \partial_w + \overline{\partial_w} + d_D$ over $W \times D$.

Proof. We have that

$$\lim_{\epsilon \to 0} {\int_{B^{0,\epsilon}}}' H^* \overline{C} = \lim_{\epsilon \to 0} {\int_{s(P_\epsilon(D'))}}' H^* \overline{C} = \lim_{\epsilon \to 0} {\int_{D'}}' P_\epsilon^* s^* H^* \overline{C}.$$

Propositions 5.5 and 5.6, and the definition of admissible singularity, show that $s^* H^* \overline{C}$ extends continuously to D', and is uniformly bounded. Therefore we can exchange limit and integral in the last integral. Hence, the integral (5.14) equals the $(1,0)$ part of (5.13) by Lemma 5.3, Propositions 5.5 and 5.6. Now we prove the integral (5.14) is finite. By equation (5.12), near $z_k \in Z$ we have

$$\begin{aligned}(c_1 + ic_{23})(z) &= -\frac{1}{2}(\,(\phi - 2i\theta) \circ f\,)'(z) \\ &= \frac{1}{2}\frac{f'(z) - f'(z_k) - f_z(z_k)(z_k)'}{z - z_k} - \frac{1}{2}(\,(\phi - \log \tilde{h}) \circ f\,)'(z).\end{aligned}$$

Note that $d_D\omega_{23}$ is a constant times the volume form and $c_1 + ic_{23}$ is singular at Z by the above equality, but its wedge product with the volume form is integrable. For the second term, we use the following formula,

$$\begin{aligned}&d_D(c_1 + ic_{23}) \wedge dw + i\partial_w \omega_{23} \\ &\quad = -(\phi'_{\bar\zeta} \circ f \overline{f}_z + \phi_{\zeta\bar\zeta} \circ f \overline{f}_z f')dz \wedge dw - (\phi'_{\bar\zeta} \circ f \overline{f}_{\bar z} + \phi_{\zeta\bar\zeta} \circ f \overline{f}_{\bar z} f')d\bar z \wedge dw\end{aligned}$$

where $\zeta = f(z)$, which can be derived from Propositions 5.5 and 5.6. Although ω_{23} is singular at Z, its wedge product with the expression above is integrable. This shows that the integral (5.14) is finite, hence $(1,0)$ part of (5.13) is finite. Similarly, the $(0,1)$ part of (5.13) is equal to the complex conjugate of (5.14) and is therefore also finite. $\square$

5.4. **Holomorphic variation of $\overline{\mathbb{CS}}$.** We begin this subsection with

Proposition 5.8. *Over $W \subset \mathbb{C}$, we have*

$$\begin{aligned}d\,(u^* \overline{\mathbb{CS}}) = {\int_{B^0}}' H^* \overline{C} &- \frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^1} \sigma^*(\theta_1 + i\theta_{23})\big|_y \\ &+ \frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^2} \sigma^*(\theta_1 + i\theta_{23})\big|_y + \sum_{r=1}^{g} {\int_{T_r - L_r T_r}}' H^* \overline{C}.\end{aligned}$$

Here the sums over $\partial\mathcal{L}^1$ and $\partial\mathcal{L}^2$ are taken with signs inherited from the orientations on $\mathcal{L}^1$ and $\mathcal{L}^2$.

Proof. First, note that $\lim_{\epsilon\to 0} d_W(u^*\mathbb{CS}^\epsilon) = d_W(u^*\mathbb{CS})$ since the diverging term $\frac{2}{\pi}(1-g)\log\epsilon$ in Definition 4.8 vanishes under d_W. By Proposition 5.7, the partial integral over $B^{0,\epsilon}$ in converges to a finite limit as $\epsilon\to 0$. By Lemma 5.2 and a similar analysis in the proof of Proposition 5.6, the right hand side of the equality in Lemma 5.2 also converges as $\epsilon\to 0$. Hence this is also true for the last terms in (5.2) given by the sum of the partial integrals over $(T_r^\epsilon - L_rT_r^\epsilon)$. Taking $\epsilon\to 0$ on both sides of (5.2), and using Lemmas 5.1 and 5.2 we have the result. □

The remainder of this section is devoted to finding an explicit expression for $d\,\overline{\mathbb{CS}}(\varpi)$ in the case that $\varpi\in T^{1,0}U$ at u_0 is a holomorphic tangent vector. The final result is given in Theorem 5.14.

Lemma 5.9. *For the holomorphic curve $u : W\to U$, we have*

$$\sigma^*(\theta_1+i\theta_{23})\big|_{y_j(0)} - \sigma^*(\theta_1+i\theta_{23})\big|_{L_{r(j)}(u(0))y_j(0)} = -(L_{r(j)}\circ u)^*(\theta_1+i\theta_{23})$$

where $(L_{r(j)}\circ u)^*(\theta_1+i\theta_{23})$ is a $(0,1)$-form on W for $j\in\mathcal{J}$.

Proof. For brevity we write $L_{r(j)}(w) := L_{r(j)}(u(w))$ and $y_j := y_j(0)$. The map $w\mapsto \sigma(w, L_{r(j)}(0)y_j) = L_{r(j)}(w)\sigma(w,y_j)$ is the composition of the maps

$$W \xrightarrow{L_{r(j)}\times\sigma(y_j)} PSL_2(\mathbb{C})\times PSL_2(\mathbb{C}) \xrightarrow{G} PSL_2(\mathbb{C})$$

where G denotes the multiplication map. Since $\theta_1+i\theta_{23}$ is a bi-invariant 1-form on $PSL_2(\mathbb{C})$, we obtain $G^*(\theta_1+i\theta_{23}) = p_1^*(\theta_1+i\theta_{23}) + p_2^*(\theta_1+i\theta_{23})$ where p_i denotes the projection onto i-th factor $PSL_2(\mathbb{C})$. It follows that

$$\sigma(L_{r(j)}(0)y_j)^*(\theta_1+i\theta_{23}) = ((L_{r(j)}\circ u)\sigma(y_j))^*(\theta_1+i\theta_{23}) = (L_{r(j)}\circ u)^*(\theta_1+i\theta_{23})+\sigma(y_j)^*(\theta_1+i\theta_{23}).$$

Hence,

$$\sigma^*(\theta_1+i\theta_{23})\big|_{y_j} - \sigma^*(\theta_1+i\theta_{23})\big|_{L_{r(j)}(0)y_j} = -(L_{r(j)}\circ u)^*(\theta_1+i\theta_{23}).$$

Since $L_{r(j)}\circ u : W\to PSL_2(\mathbb{C})$ is a holomorphic map, and $\theta_1+i\theta_{23}$ is a $(0,1)$-form on $PSL_2(\mathbb{C})$ (see the section 3 of [18]), the statement follows. □

Lemma 5.10. *The partial integral $\sum_{r=1}^g \int'_{T_r-L_rT_r} H^*\overline{C}$ is a $(0,1)$-form over W.*

Proof. For each $w\in W$ and $x\in D^r$,

$$\begin{aligned} H(w, s_0(L_r(0)x)) &= s(u(w), \mathbf{f}(w, L_r(0)x)) \\ &= s(u(w), L_r(w)\mathbf{f}(w,x)) = L_r(w)s(u(w),\mathbf{f}(w,x)) = L_r(w)H(w,s_0(x)), \end{aligned}$$

where $L_r(w) := L_r(u(w))$. Hence $H : W\times L_rT_r\to PSL_2(\mathbb{C})$ can be considered as the composition of the maps

$$W\times T_r \xrightarrow{L_r\times H} PSL_2(\mathbb{C})\times PSL_2(\mathbb{C}) \xrightarrow{G} PSL_2(\mathbb{C})$$

where $(L_r\times H)(w, s_0(x)) = (L_r(w), H(w, s_0(x)))$ and G denotes the multiplication map. The pull back of $\overline{C}$ by G is given by $G^*\overline{C} = p_1^*\overline{C} + (G^*\overline{C})^{2,1} + (G^*\overline{C})^{1,2} + p_2^*\overline{C}$, where $p_i : PSL_2(\mathbb{C})\times PSL_2(\mathbb{C})\to PSL_2(\mathbb{C})$, $i=1,2$, are the projections from the two factors, and where superscripts on a form indicate the degree in the two factors. Taking the pull back of $G^*\overline{C}$ by $L_r \times H$, we have

$$(G(L_r \times H))^*\overline{C} = L_r^*\overline{C} + (L_r \times H)^*(G^*\overline{C})^{2,1} + (L_r \times H)^*(G^*\overline{C})^{1,2} + H^*\overline{C}.$$

Hence we have the following equality for the partial integrals:

$$\int_{T^r}' H^*\overline{C} - \int_{L_r T^r}' H^*\overline{C} = -\int_{T^r}' (L_r \times H)^*(G^*\overline{C})^{1,2}.$$

Since the map $w \in W \mapsto L_r(w) \in PSL_2(\mathbb{C})$ is holomorphic, the dw term in $(L_r \times H)^*(G^*\overline{C})^{1,2}$ vanishes under the above partial integration. Hence the 1-form on W obtained by the partial integration of $\int_{T^r - L_r T^r}' H^*\overline{C}$ does not involve dw, that is, it is of type $(0, 1)$. $\square$

From now on, $\dot{}$ will denote the derivative with respect to w at $w = 0$, for instance, $\dot{\phi} = \frac{\partial}{\partial w}\big|_{w=0}\phi$. By the results on varying the hyperbolic metric in [1], we have, for all $z \in \Omega$,

$$\dot{\phi} + \phi_z \dot{f} + \dot{f}_z = 0. \tag{5.15}$$

(The same is true for the flat metric of area 1 in the case that the genus of X is 1.) From this, we also have

$$\dot{\phi}_z + \phi_{zz}\dot{f} + \phi_z\dot{f}_z + \dot{f}_{zz} = 0, \qquad \dot{\phi}_{\bar z} + \phi_{z\bar z}\dot{f} + \phi_z\dot{f}_{\bar z} + \dot{f}_{z\bar z} = 0. \tag{5.16}$$

Since $2i\theta = \log h - \log \bar{h}$,

$$2i\theta_z = \frac{h_z}{h}, \qquad 2i\theta_{\bar z} = -\frac{\bar{h}_{\bar z}}{\bar{h}}, \qquad \theta_{z\bar z} = 0. \tag{5.17}$$

Since Φ is a holomorphic family, we also have

$$\dot{\theta}_{\bar z} = 0. \tag{5.18}$$

It will be convenient in what follows to make the definition $\psi := \phi - 2i\theta$.

Lemma 5.11. *The following terms are invariant under the action of* Γ,

$$-\dot{f}_z - (2i\theta)^{\cdot} - (2i\theta)_z\dot{f} = \dot{\psi} + \psi_z\dot{f}.$$

Proof. The equality follows from $\dot{\phi} + \phi_z\dot{f} + \dot{f}_z = 0$. To see the invariance under the action of Γ, we note

$$(\phi - 2i\theta)^{\cdot}(z) = (\phi - 2i\theta)^{\cdot}(\gamma z) + (\phi - 2i\theta)_z(\gamma z)\dot{\gamma}(z),$$
$$(\phi - 2i\theta)_z(z) = (\phi - 2i\theta)_z(\gamma z)\gamma_z(z),$$

which follow from (5.5) and (5.7). Combining these and $\dot{f} \circ \gamma = \dot{\gamma} + \gamma_z\dot{f}$ completes the proof. $\square$

From now on, for convenience, we abbreviate $z_k(0)$ to z_k, and $\dot{z}_k(0)$ to $\dot{z}_k$.

Proposition 5.12. *For* $\varpi \in T^{1,0}U$ *at* $u_0 \in U$, *we have*

$$\begin{aligned}\partial\,\overline{\mathbb{CS}}(\varpi) = \frac{1}{4\pi^2}\int_D d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge (d(c_1 + ic_{23}) - i\dot{\omega}_{23})\\ - \frac{1}{4\pi}\sum_{z_k \in Z}\Big(\dot{f}_z + \frac{1}{2}(\log\tilde{h}_k)^{\cdot} + \frac{1}{2}(\log\tilde{h}_k)_z\dot{f} - (\phi - \frac{1}{2}\log\tilde{h}_k)_z f_z\dot{z}_k\Big)(z_k)\end{aligned}$$

where Z denotes the set of zeros of Φ in the fundamental domain D of Γ and $\tilde{h}_k$ is defined by equation (5.12).

Proof. By Proposition 5.8 and Lemma 5.9, $\partial\,\overline{\mathbb{CS}}(\varpi)$ is equal to the evaluation of the one form

$$\int_{B_0}' H^*\overline{C} + n(j)\frac{1}{2\pi}\sum_{j\in\mathcal{J}}(L_{r(j)}\circ u)^*(\theta_1 + i\theta_{23})$$
$$+\frac{1}{2\pi}\sum_{y\in(\partial\mathcal{L}^2\cap D)}\sigma^*(\theta_1+i\theta_{23})\big|_y + \sum_{r=1}^{g}\int_{(T_r-L_rT_r)}' H^*\overline{C}$$

on $\frac{\partial}{\partial w}$. Here $n(j)$ is the index of the singularity at the corresponding component of $\mathcal{L}$, so $n(j)=1$ or -1 if the points $y_j(0)$, $L_{r(j)}(0)y_j(0)$ are in $\partial\mathcal{L}^1$ or $\partial\mathcal{L}^2$ respectively. By Lemma 5.3 and Proposition 5.7, the evaluation of the first term on $\frac{\partial}{\partial w}$ is given by

$$\frac{1}{4\pi^2}\int_D d\omega_{23}\wedge(c_1+ic_{23}) + \omega_{23}\wedge\big(d(c_1+ic_{23}) - i\dot{\omega}_{23}\big).$$

The second and fourth terms vanish on $\frac{\partial}{\partial w}$, since they are $(0,1)$-forms by Lemmas 5.9 and 5.10. Using Lemma 5.2, and following the proof of Proposition 5.6, we find that the third term evaluated on $\frac{\partial}{\partial w}$ is given by

$$\begin{aligned}
&\Big(\frac{1}{2\pi}\sum_{y\in(\partial\mathcal{L}^2\cap D)}\sigma^*(\theta_1+i\theta_{23})\big|_y\Big)\Big(\frac{\partial}{\partial w}\Big)\\
=&\frac{1}{4\pi}\sum_{z_k\in Z}\big(\,(\dot\phi - i\tilde{\theta}^{\cdot}) + (\phi - i\tilde\theta)_z\dot f + (\phi-i\tilde\theta)_z f_z\dot z_k\,\big)(z_k)\\
=&\frac{1}{4\pi}\sum_{z_k\in Z}\big(-\dot f_z - \frac{1}{2}(\log\tilde h_k)^{\cdot} - \frac{1}{2}(\log\tilde h_k)_z\dot f + (\phi - \frac{1}{2}\log\tilde h_k)_z f_z\dot z_k\,\big)(z_k).
\end{aligned}$$

Here the last equality follows from (5.8) and (5.15). This completes the proof. □

Proposition 5.13. *The following equality holds:*

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_D d\omega_{23}\wedge(c_1+ic_{23}) + \omega_{23}\wedge(d(c_1+ic_{23}) - i\dot\omega_{23})\\
=&-\frac{1}{2\pi^2}\lim_{\delta\to 0}\int_{D_\delta}(\phi_{zz} - \frac{1}{2}\phi_z^2 - 2\theta_z^2 - 2i\theta_{zz})\mu\; d^2z\\
&-\frac{1}{4\pi}\sum_{z_k\in Z}\big(2\dot f_z + (\log\tilde h_k)^{\cdot} + (\log\tilde h_k)_z\dot f + (\phi - \log\tilde h_k)_z f_z\dot z_k\,\big)(z_k),
\end{aligned}$$

where D_δ is a subset of D whose δ-open neighborhoods of Z are removed and $d^2z = \frac{i}{2}dz\wedge d\bar z$.

Note that, since circles are preserved under holomorphic change of coordinates, the limit as $\delta\to 0$ is independent of the choice of local coordinates.

Proof. By Proposition 5.6,

$$\begin{aligned}
c_1 + ic_{23} &= -\frac{1}{2}\big(\,\psi\circ f\,\big)^{\cdot}\\
&= -\frac{1}{2}(\dot\phi + \phi_z\dot f - 2i\dot\theta - 2i\theta_z\dot f) = (i\dot\theta + i\theta_z\dot f + \frac{1}{2}\dot f_z)
\end{aligned}\tag{5.19}$$

where we used (5.15) for the third equality. From (5.19), we can also derive

$$d(c_1 + ic_{23}) = -\frac{1}{2}\big(\,(\dot\psi_z + \psi_{zz}\dot f + \psi_z \dot f_z)dz + (\dot\psi_{\bar z} + \psi_{z\bar z}\dot f + \psi_z \dot f_{\bar z})d\bar z\,\big). \tag{5.20}$$

By Proposition 5.5,

$$-i\dot\omega_{23} = \frac{1}{2}\big(\,(\dot\psi_z + \psi_{zz}\dot f + \psi_z \dot f_z)dz + (-\dot{\bar\psi}_{\bar z} - \bar\psi_{z\bar z}\dot f + \psi_z \dot f_{\bar z})d\bar z\,\big). \tag{5.21}$$

Again by Proposition 5.5,

$$d\omega_{23} = -i\phi_{z\bar z} dz \wedge d\bar z = -i\psi_{z\bar z} dz \wedge d\bar z.$$

Combining this and (5.19), (5.20), (5.21), we get

$$d\omega_{23} \wedge (c_1 + ic_{23}) = -i\psi_{z\bar z}(i\dot\theta + i\theta_z \dot f + \frac{1}{2}\dot f_z) dz \wedge d\bar z,$$

which is an invariant $(1,1)$-form under the action of Γ by Lemma 5.11 and

$$\omega_{23} \wedge (d(c_1 + c_{23}) - i\dot\omega_{23}) = -\frac{i}{2}\psi_z(\dot\phi_{\bar z} + \phi_{z\bar z}\dot f) dz \wedge d\bar z = \frac{i}{2}\psi_z(\phi_z \dot f_{\bar z} + \dot f_{z\bar z}) dz \wedge d\bar z$$

where we used (5.16) for the last equality.

By the above equalities,

$$\begin{aligned}
&\int_{D_\delta} d\omega_{23} \wedge (c_1 + ic_{23}) = -i\int_{D_\delta} \psi_{z\bar z}(i\dot\theta + i\theta_z \dot f + \frac{1}{2}\dot f_z)\ dz \wedge d\bar z\\
=&i\int_{D_\delta} \psi_z(i\theta_z \dot f_{\bar z} + \frac{1}{2}\dot f_{z\bar z})\ dz \wedge d\bar z + i\int_{\partial D_\delta} \psi_z(i\dot\theta + i\theta_z \dot f + \frac{1}{2}\dot f_z)dz\\
=&i\int_{D_\delta} (\psi_z i\theta_z - \frac{1}{2}\psi_{zz})\dot f_{\bar z}\ dz \wedge d\bar z + i\int_{\partial D_\delta} \psi_z(i\dot\theta + i\theta_z \dot f + \frac{1}{2}\dot f_z)dz + \frac{1}{2}\psi_z \dot f_{\bar z} d\bar z
\end{aligned} \tag{5.22}$$

where ∂D_δ has the induced orientation from D_δ. In the integral over ∂D_δ, the contributions from C^r and $-L_r(0)C^r$ cancel, since the integrands concerned are invariant. We also have

$$\begin{aligned}
&\int_{D_\delta} \omega_{23} \wedge (d(c_1 + c_{23}) - i\dot\omega_{23}) = \frac{i}{2}\int_{D_\delta} \psi_z(\phi_z \dot f_{\bar z} + \dot f_{z\bar z})\ dz \wedge d\bar z\\
=&\frac{i}{2}\int_{D_\delta} (\psi_z\phi_z - \psi_{zz})\dot f_{\bar z}\ dz \wedge d\bar z + \frac{i}{2}\int_{\partial D_\delta} \psi_z \dot f_{\bar z} d\bar z,
\end{aligned} \tag{5.23}$$

where once again the contributions from C^r and $-L_r(0)C^r$ cancel in the integral over ∂D_δ. By (5.19), (5.22) and (5.23),

$$\begin{aligned}
&\int_{D_\delta} d\omega_{23} \wedge (c_1 + ic_{23}) + \omega_{23} \wedge (d(c_1 + ic_{23}) - i\dot\omega_{23})\\
=&\frac{i}{2}\int_{D_\delta} (\psi_z\bar\psi_z - 2\psi_{zz})\dot f_{\bar z}\ dz \wedge d\bar z - \frac{i}{2}\int_{\partial D_\delta} \psi_z(\,\psi \circ f\,)^{\cdot} dz + i\int_{\partial D_\delta} \psi_z \dot f_{\bar z} d\bar z.
\end{aligned} \tag{5.24}$$

For the last integral on the right hand side of (5.24), we have

$$\begin{aligned}
i\int_{\partial D_\delta} \psi_z \dot f_{\bar z}\, d\bar z =& -i\sum_{z_k \in Z}\int_{|z-z_k|=\delta} \big(-\frac{1}{z-z_k} + (\phi - \log \tilde h_k)_z\,\big)\, \dot f_{\bar z}\, d\bar z\\
=& -i\sum_{z_k \in Z}\int_{|z-z_k|=\delta} -\frac{1}{z-z_k}\, \dot f_{\bar z}\, d\bar z + O(\delta) = O(\delta).
\end{aligned} \tag{5.25}$$

To analyze the second integral on the right hand side of (5.24), we use (5.12). This implies that, near $z_k \in Z$, we have

$$((\phi - 2i\theta)\circ f\,)^{\cdot}(z) = -\frac{\dot{f}(z) - \dot{f}(z_k) - f_z(z_k)\dot{z}_k}{z - z_k} + ((\phi - \log\tilde{h}_k)\circ f\,)^{\cdot}(z). \tag{5.26}$$

Therefore, we can rewrite the second integral of (5.24) as

$$\begin{aligned}
&-\frac{i}{2}\int_{\partial D_\delta}\psi_z(\,(\phi-2i\theta)\circ f\,)^{\cdot}dz\\
=&\frac{i}{2}\sum_{z_k\in Z}\Big(\int_{|z-z_k|=\delta}(-\frac{1}{z-z_k}+(\phi-\log\tilde{h}_k)_z)\,\big(-\frac{\dot{f}(z)-\dot{f}(z_k)-f_z(z_k)\dot{z}_k}{z-z_k}\big)\,dz\\
&\qquad+\int_{|z-z_k|=\delta}(-\frac{1}{z-z_k}+(\phi-\log\tilde{h}_k)_z)\,\big((\phi-\log\tilde{h}_k)\circ f\,)^{\cdot}(z)\big)\,dz\Big)\\
=&-\pi\sum_{z_k\in Z}\big(\dot{f}_z(z_k)+(\phi-\log\tilde{h}_k)_z(z_k)f_z(z_k)\dot{z}_k-((\phi-\log\tilde{h}_k)\circ f\,)^{\cdot}(z_k)\big)+O(\delta)\\
=&-\pi\sum_{z_k\in Z}\big(2\dot{f}_z+(\log\tilde{h}_k)^{\cdot}+(\log\tilde{h}_k)_z\dot{f}+(\phi-\log\tilde{h}_k)_z f_z\dot{z}_k\big)(z_k)+O(\delta).
\end{aligned}$$

Combining this with (5.24) and (5.25), we conclude

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{D_\delta} d\omega_{23}\wedge(c_1+ic_{23})+\omega_{23}\wedge(d(c_1+ic_{23})-i\dot{\omega}_{23})\\
=&\lim_{\delta\to 0}\int_{D_\delta}(\phi_z^2-2\phi_{zz}+4\theta_z^2+4i\theta_{zz})\dot{f}_{\bar{z}}\;d^2z\\
&-\pi\sum_{z_k\in Z}\big(2\dot{f}_z+(\log\tilde{h}_k)^{\cdot}+(\log\tilde{h}_k)_z\dot{f}+(\phi-\log\tilde{h}_k)_z f_z\dot{z}_k\big)(z_k).
\end{aligned}$$

Recalling that $\dot{f}_{\bar{z}} = \mu$ completes the proof. □

Note that we have the formulas

$$\mathcal{S}(J^{-1}) = \phi_{zz} - \frac{1}{2}\phi_z^2, \qquad \mathcal{S}(h_\Phi) = \frac{h_{zz}}{h} - \frac{3}{2}\frac{h_z^2}{h^2} = 2\theta_z^2 + 2i\theta_{zz}$$

where $\mathcal{S}$ denotes the Schwarzian derivative, $J : H^2 \to \Omega$ is the universal covering map of Ω, (or $J : \mathbb{C} \to \Omega$ in the case of genus 1), and h_Φ is a multi-valued function such that $dh_\Phi = \Phi$. By these formulas and Propositions 5.12 and 5.13, we have the following theorem.

Theorem 5.14. *For $\varpi \in T^{1,0}U$ at $u_0 \in U$, and the corresponding $\mu \in \mathcal{H}^{-1,1}(\Omega, \Gamma)$,*

$$\begin{aligned}
\partial\,\overline{\mathbb{CS}}(\varpi) =& -\frac{1}{2\pi^2}\lim_{\delta\to 0}\int_{D_\delta}\big(\mathcal{S}(J^{-1})-\mathcal{S}(h_\Phi)\big)\mu\,d^2z\\
&-\frac{1}{4\pi}\sum_{z_k\in Z}\big(3\dot{f}_z+\frac{3}{2}(\log\tilde{h}_k)^{\cdot}+\frac{3}{2}(\log\tilde{h}_k)_z\dot{f}-\frac{1}{2}(\log\tilde{h}_k)_z f_z\dot{z}_k\big)(z_k).
\end{aligned}$$

Corollary 5.15. *For $\varpi \in T^{1,0}U$ at $u_0 \in U$, and the corresponding $\mu \in \mathcal{H}^{-1,1}(\Omega,\Gamma)$,*

$$\partial\, CS(\varpi) = \frac{i}{4\pi^2}\lim_{\delta\to 0}\int_{D_\delta}\mathcal{S}(h_\Phi)\mu\, d^2z \\ -\frac{i}{8\pi}\sum_{z_k\in Z}\big(\, 3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z \dot{z}_k\,\big)(z_k).$$

Proof. This follows from directly from Theorem 5.14, since we have

$$\mathbb{CS}(M_X, s_\Phi) = \frac{1}{\pi^2}W(M) + 2iCS(M_X, s_\Phi),$$

and, by [13], [21], it is known that $\partial W = \frac{1}{4}\phi_z^2 - \frac{1}{2}\phi_{zz} = -\frac{1}{2}\mathcal{S}(J^{-1})$. □

6. Regularized Polyakov integral over X

In this section, we introduce an regularized integral defined in terms of the metric g_X and the holomorphic 1-form Φ over X. We assume that g_X is the hyperbolic metric if the genus of X is greater than 1, and the flat metric of area 1 if the genus equals 1. We assume that Φ has only simple zeroes and we denote by Z its zero set.

Now we define

$$I(X,\Phi) = \lim_{\delta\to 0}\Big(\int_{X_\delta}|\psi_z|^2\, d^2z + \frac{i}{2}\sum_{p_k\in Z}\int_{S_\delta(z_k)}\frac{(\phi - 2\log|h|)(z)}{\bar{z}-\bar{z}_k}d\bar{z}\,\Big) \\ - \pi\sum_{p_k\in Z}(\phi - \log|\tilde{h}_k|)(z_k). \tag{6.1}$$

Here z, in the integral around p_k, represents a local coordinate near p_k, with $z_k = z(p_k)$. The set X_δ denotes the complement of δ-open discs $|z - z_k| < \delta$ centered at each $z_k \in Z$ in X, and $S_\delta(z_k)$ denotes a part of ∂X_δ which is the δ-circle centered at z_k with the induced orientation from X_δ. Note that each of the terms in (6.1) are independent of the choice of local coordinates, by the transformation laws given in subsection 3.2. Note also that, since circles are preserved under change of coordinates, the limit as $\delta \to 0$ is independent of the choice of local coordinates. Hence I is a well-defined function on $\mathcal{H}_g(1,\dots,1)$.

Suppose that ϖ is a tangent vector at $u_0 \in \mathcal{H}_g(1,\dots,1)$, and that U is a neighborhood of u_0. We define a corresponding curve $u : W \to U$, for $W \subset \mathbb{C}$, and a corresponding deformation map $f(w,\cdot) : X \to X_w$ for each $w \in W$, in the same way as in subsection 5.1. We also define the local coordinate expressions h, $\tilde{h}_k$ and z_k in the same way as the discussion before the equation (5.12), except that here we do not assume a global uniformization coordinate, only local coordinates near the zeroes of Φ. For convenience we abbreviate $z_k(0)$ to z_k, and $\dot{z}_k(0)$ to $\dot{z}_k$.

Theorem 6.1. *For $\varpi \in T^{1,0}\mathcal{H}_g(1,\dots,1)$ at the point (X,Φ) and the corresponding $\mu \in \mathcal{H}^{-1,1}(X)$,*

$$\partial I(\varpi) = 2\lim_{\delta\to 0}\int_{X_\delta}\big(\,\phi_{zz} - \frac{1}{2}\phi_z^2 - 2\theta_z^2 - 2i\theta_{zz}\,\big)\mu\, d^2z \\ +\pi\sum_{p_k\in Z}\Big(3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z\dot{z}_k\Big)(z_k).$$

Here $\phi_{zz} - \frac{1}{2}\phi_z^2 - 2\theta_z^2 - 2i\theta_{zz}$ is a meromorphic quadratic differential over X.

Proof. The domain $X_{w,\delta}$ is given by deleting the δ-discs centered at the $f(w,z_k(w))$ for $z_k\in Z$. Its boundaries are given by the circles $S_\delta(f(w,z_k(w)))$. Now we consider the pre-image domain, denoted by the same notation, of $X_{w,\delta}$ by f_w in X which has boundaries denoted by $B_\delta(z_k(w))$. Let us take δ_0 such that the δ_0-disc centered at z_k contains $B_\delta(z_k(w))$ for each $z_k\in Z$, and take w in an open neighborhood W of the origin in $\mathbb{C}$. Then $X_{w,\delta}$ in X decomposes into $X_{\delta_0}\cup A_{\delta_0,\delta}$. Here $A_{\delta_0,\delta}=\cup_{z_k\in Z}A_{\delta_0,\delta}(z_k)$ where the region $A_{\delta_0,\delta}(z_k)$ has two boundaries $S_{\delta_0}(z_k)$ and $B_\delta(z_k(w))$.

For the integral $|\psi_z|^2d^2z$ over $A_{\delta_0,\delta}$, we have

$$
\begin{aligned}
&\int_{A_{\delta_0,\delta(z_k)}}|\psi_z|^2d^2z\\
=&\int_{A_{\delta_0,\delta(z_k)}}|(\phi-\log\tilde h_k)_z|^2\,d^2z-\int_{A_{\delta_0,\delta(z_k)}}\frac{(\phi-\log\bar{\tilde h}_k)_{\bar z}}{z-z_k(w)}\,d^2z-\int_{A_{\delta_0,\delta(z_k)}}\frac{(\phi-\log h)_z}{\bar z-\bar z_k(w)}\,d^2z\\
=&\int_{A_{\delta_0,\delta(z_k)}}|(\phi-\log\tilde h_k)_z|^2\,d^2z+\frac{i}{2}\int_{\partial A_{\delta_0,\delta(z_k)}}\frac{(\phi-\log\bar{\tilde h}_k)}{z-z_k(w)}\,dz-\frac{i}{2}\int_{\partial A_{\delta_0,\delta(z_k)}}\frac{(\phi-2\log|h|)}{\bar z-\bar z_k(w)}\,d\bar z.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{X_{w,\delta}}|\psi_z|^2\,d^2z+\frac{i}{2}\sum_{p_k\in Z}\int_{B_\delta(z_k(w))}\frac{(\phi-2\log|h|)}{\bar z-\bar z_k(w)}d\bar z\\
=&\int_{X_{\delta_0}}|\psi_z|^2\,d^2z+\int_{A_{\delta_0,\delta}}|(\phi-\log\tilde h_k)_z|^2\,d^2z+\frac{i}{2}\int_{\partial A_{\delta_0,\delta}}\frac{(\phi-\log\bar{\tilde h}_k)}{z-z_k(w)}\,dz\\
&+\frac{i}{2}\sum_{p_k\in Z}\int_{S_{\delta_0}(z_k)}\frac{(\phi-2\log|h|)}{\bar z-\bar z_k(w)}\,d\bar z
\end{aligned}
\tag{6.2}
$$

where $B_\delta(z_k(w))$ and $S_{\delta_0}(z_k)$ have the orientation induced from $A_{\delta_0,\delta}(z_k)$ and X_{δ_0} respectively.

Now, we consider the holomorphic variation of each of the terms on the right hand side of (6.2). First, we deal with the term $I_{\delta_0}=\int_{X_{\delta_0}}|\psi_z|^2\,d^2z$. For this, observe that

$$
\begin{aligned}
\delta_\mu\big(\psi_z dz\big)&=\big(\dot\psi_z+\psi_{zz}\dot f\big)dz+\psi_z(\dot f_z dz+\dot f_{\bar z}d\bar z),\\
\delta_\mu\big(\bar\psi_{\bar z}d\bar z\big)&=\big(\dot{\bar\psi}_{\bar z}+\bar\psi_{\bar z z}\dot f\big)d\bar z=\big(\dot\phi_{\bar z}+\phi_{\bar z z}\dot f\big)d\bar z.
\end{aligned}
$$

Here, δ_μ denotes the Lie derivative. See Section 2.3 of [15] for details. Combining these facts with (5.16) and Lemma 5.11, we have

$$
\begin{aligned}
\partial I_{\delta_0}(\varpi)=&-\frac{i}{2}\int_{X_{\delta_0}}\psi_z(\phi_z\dot f_{\bar z}+\dot f_{z\bar z})\,dz\wedge d\bar z\\
&-\frac{i}{2}\int_{X_{\delta_0}}\bar\psi_{\bar z}(\dot f_{zz}+(2i\theta)_z^{\cdot}+((2i\theta)_z\dot f)_z)\,dz\wedge d\bar z.
\end{aligned}
\tag{6.3}
$$

Let us denote the two terms on the right hand side of (6.3) by $(\partial I_{\delta_0}(\varpi))_i$ for $i=1,2$. Recalling that $\psi_z\dot f_{\bar z}d\bar z$ is an invariant $(0,1)$-form, we have

$$
(\partial I_{\delta_0}(\varpi))_1=-\frac{i}{2}\Big(\int_{X_{\delta_0}}\psi_z\phi_z\dot f_{\bar z}\,dz\wedge d\bar z-\int_{X_{\delta_0}}\psi_{zz}\dot f_{\bar z}\,dz\wedge d\bar z+\int_{\partial X_{\delta_0}}\psi_z\dot f_{\bar z}\,d\bar z\Big)
$$

where ∂X_{δ_0} has the induced orientation from X_{δ_0}. For $(\partial I_{\delta_0}(\varpi))_2$, by Lemma 5.11 and (5.17), (5.18),

$$\begin{aligned}(\partial I_{\delta_0}(\varpi))_2 = \frac{i}{2}\Big(&\int_{X_{\delta_0}} \psi_{z\bar z}(\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ dz\wedge d\bar z \\ &\qquad - \int_{\partial X_{\delta_0}} \bar\psi_{\bar z}(\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ d\bar z\ \Big) \\ = -\frac{i}{2}\Big(&\int_{X_{\delta_0}} \psi_z(\dot f_{z\bar z} + (2i\theta)_z \dot f_{\bar z}\ )\ dz\wedge d\bar z \\ &\qquad + \int_{\partial X_{\delta_0}} (\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ \big(\psi_z dz + \bar\psi_{\bar z} d\bar z\big)\ \Big).\end{aligned}$$

Dealing with the term $\psi_z \dot f_{z\bar z}$ as before,

$$\begin{aligned}(\partial I_{\delta_0}(\varpi))_2 = -\frac{i}{2}\Big(&\int_{X_{\delta_0}} \psi_z (2i\theta)_z \dot f_{\bar z}\ dz\wedge d\bar z - \int_{X_{\delta_0}} \psi_{zz} \dot f_{\bar z}\ dz\wedge d\bar z \\ &+ \int_{\partial X_{\delta_0}} \psi_z \dot f_{\bar z}\ d\bar z + \int_{\partial X_{\delta_0}} (\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ \big(\psi_z dz + \bar\psi_{\bar z} d\bar z\big)\ \Big).\end{aligned}$$

Combining computations for $(\partial I_{\delta_0}(\varpi))_1$ and $(\partial I_{\delta_0}(\varpi))_2$, we get

$$\begin{aligned}\partial I_{\delta_0}(\varpi) = -\frac{i}{2}\Big(&\int_{X_{\delta_0}} \psi_z \bar\psi_z \dot f_{\bar z}\ dz\wedge d\bar z - 2\int_{X_{\delta_0}} \psi_{zz} \dot f_{\bar z}\ dz\wedge d\bar z \\ &+ 2\int_{\partial X_{\delta_0}} \psi_z \dot f_{\bar z}\ d\bar z + \int_{\partial X_{\delta_0}} (\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ \big(\psi_z dz + \bar\psi_{\bar z} d\bar z\big)\ \Big).\end{aligned} \tag{6.4}$$

Now let us deal with the integrals over ∂X_{δ_0}. First, by (5.25) we have

$$\int_{\partial X_{\delta_0}} \psi_z \dot f_{\bar z}\ d\bar z = \int_{\partial X_{\delta_0}} (\phi - 2i\theta)_z \dot f_{\bar z}\ d\bar z = O(\delta_0). \tag{6.5}$$

For the other boundary integral given in the last line of (6.4), using (5.26), near $p_k \in Z$ we have

$$\begin{aligned}&-(\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ \big((\phi - 2i\theta)_z dz + (\phi + 2i\theta)_{\bar z} d\bar z\big) \\ =&\big(\ (\phi - 2i\theta)\circ f\ \big)^{\cdot}\ \big((\phi - 2i\theta)_z dz + (\phi + 2i\theta)_{\bar z} d\bar z\big) \\ =&\Big(\ -\frac{\dot f(z) - \dot f(z_k) - f_z(z_k)\dot z_k}{z - z_k} + ((\phi - \log \tilde h_k)\circ f)^{\cdot}\Big) \\ &\cdot\ \Big((-\frac{1}{z - z_k} + (\phi - \log\tilde h_k)_z)dz + (-\frac{1}{\bar z - \bar z_k} + (\phi - \log\bar{\tilde h}_k)_{\bar z})d\bar z\ \Big).\end{aligned}$$

Using this and some computation as before, we obtain

$$
\begin{aligned}
& -\frac{i}{2}\int_{\partial X_{\delta_0}} (\dot f_z + (2i\theta)^{\cdot} + (2i\theta)_z \dot f)\ \big((\phi-2i\theta)_z dz + (\phi+2i\theta)_{\bar z} d\bar z\big) \\
&= -\frac{i}{2}\sum_{p_k\in Z}\Big(\int_{|z-z_k|=\delta_0}\big(-\frac{\dot f(z)-\dot f(z_k)-f_z(z_k)\dot z_k}{z-z_k}\big)\big(-\frac{1}{z-z_k}dz - \frac{1}{\bar z-\bar z_k}d\bar z\big) \\
&\qquad\qquad + \big(-\frac{\dot f(z)-\dot f(z_k)-f_z(z_k)\dot z_k}{z-z_k}\big)\big((\phi-\log\tilde h_k)_z dz + (\phi-\log\bar{\tilde h}_k)_{\bar z} d\bar z\big) \\
&\qquad\qquad + ((\phi-\log\tilde h_k)\circ f)^{\cdot}(z)\big(-\frac{1}{z-z_k}dz - \frac{1}{\bar z-\bar z_k}d\bar z\big)\Big) + O(\delta_0) \\
&= \pi\sum_{p_k\in Z}(\phi-\log\tilde h_k)_z(z_k) f_z(z_k)\dot z_k + O(\delta_0).
\end{aligned}
\tag{6.6}
$$

By (6.4), (6.5) and (6.6),

$$
\begin{aligned}
&\partial\Big(\int_{X_{\delta_0}}|\psi_z|^2\, d^2z\Big)(\varpi) \\
&= \int_{X_{\delta_0}}(2\phi_{zz}-\phi_z^2-4\theta_z^2-4i\theta_{zz})\dot f_{\bar z}\ d^2z + \pi\sum_{p_k\in Z}(\phi-\log\tilde h_k)_z(z_k)f_z(z_k)\dot z_k + O(\delta_0).
\end{aligned}
\tag{6.7}
$$

The holomorphic variation of the second term $\int_{A_{\delta_0,\delta}}|(\phi-\log\tilde h_k)_z|^2\, d^2z$ on the right hand side of (6.2) can be analyzed as above, but the integrand $|(\phi-\log\tilde h_k)_z|^2$ is regular over $A_{\delta_0,\delta}$ for any $\delta>0$. Hence, we can see that

$$
\partial\Big(\lim_{\delta\to 0}\int_{A_{\delta_0,\delta}}|(\phi-\log\tilde h_k)_z|^2\, d^2z\Big)(\varpi) = O(\delta_0).
\tag{6.8}
$$

The limit of the third term on the right hand side of (6.2) as $\delta\to 0$ is given by

$$
\sum_{p_k\in Z}\frac{i}{2}\int_{|z-z_k|=\delta_0}\frac{(\phi-\log\bar{\tilde h}_k)}{z-z_k(w)}\, dz + \pi(\phi_w-\log\bar{\tilde h}_{k,w})(f(w,z_k(w)))
\tag{6.9}
$$

where ϕ_w, $\tilde h_{k,w}$ denote (local) functions over X_w. For the holomorphic variation of the first term in (6.9), we have

$$
\begin{aligned}
&\partial\Big(\frac{i}{2}\int_{|z-z_k|=\delta_0}\frac{(\phi-\log\bar{\tilde h}_k)}{z-z_k(w)}\, dz\Big)(\varpi) \\
=&\frac{i}{2}\int_{|z-z_k|=\delta_0} -\frac{\dot f(z)-\dot f(z_k)-f_z(z_k)\dot z_k}{(z-z_k)^2}(\phi-\log\bar{\tilde h}_k)\, dz \\
&\qquad + \frac{\dot\phi+\phi_z\dot f}{z-z_k}\, dz + \frac{\phi-\log\bar{\tilde h}_k}{z-z_k}(\dot f_z dz + \dot f_{\bar z} d\bar z) \\
=& -\pi(\dot\phi+\phi_z\dot f+\phi_z f_z\dot z_k)(z_k) + O(\delta_0).
\end{aligned}
$$

For the second term in (6.9), we have

$$
\partial\Big(\pi(\phi_w-\log\bar{\tilde h}_{k,w})\big(f\big(w,z_k(w)\big)\big)\Big)(\varpi) = \pi(\dot\phi+\phi_z\dot f+\phi_z f_z\dot z_k)(z_k).
$$

Hence,

$$\partial\Big(\lim_{\delta\to 0} \frac{i}{2} \int_{\partial A_{\delta_0,\delta}} \frac{(\phi - \log \bar{\tilde{h}}_k)}{z - z_k(w)}\, dz \Big)(\varpi) = O(\delta_0). \tag{6.10}$$

In a similar way, we can show the following equality for the fourth term on the right hand side of (6.2),

$$\begin{aligned}
&\partial\Big(\frac{i}{2} \sum_{p_k\in Z} \int_{S_{\delta_0}(z_k)} \frac{(\phi - 2\log|h|)}{\bar z - \bar z_k(w)}\, d\bar z \Big)(\varpi) \\
=&\pi \sum_{p_k\in Z} (\dot f_z - (\phi - \log \tilde h_k)^{\cdot} - (\phi - \log\tilde h_k)_z \dot f)(z_k) + O(\delta_0).
\end{aligned} \tag{6.11}$$

Combining the equalities (6.7), (6.8), (6.10), and (6.11), we have

$$\begin{aligned}
&\partial\Big(\lim_{\delta\to 0} \big(\int_{X_{w,\delta}} |\psi_z|^2\, d^2 z + \frac{i}{2} \sum_{p_k\in Z} \int_{B_\delta(z_k(w))} \frac{(\phi - 2\log|h|)(z)}{\bar z - \bar z_k(w)} d\bar z \big) \Big)(\varpi) \\
=& \lim_{\delta_0\to 0} \int_{X_{\delta_0}} (2\phi_{zz} - \phi_z^2 - 4\theta_z^2 - 4i\theta_{zz}) \dot f_{\bar z}\ d^2 z \\
&+ \pi \sum_{p_k\in Z} (\dot f_z - (\phi - \log\tilde h_k)^{\cdot} - (\phi - \log\tilde h_k)_z \dot f + (\phi - \log\tilde h_k)_z f_z \dot z_k)(z_k).
\end{aligned} \tag{6.12}$$

Finally combining (6.2), (6.12) and the following equality

$$\begin{aligned}
&\partial\Big(-\pi \sum_{p_k\in Z} (\phi - \log|\tilde h_k|)(z_k) \Big)(\varpi) \\
&\qquad = -\pi \sum_{p_k\in Z} \big((\phi - \frac{1}{2}\log\tilde h_k)^{\cdot} + (\phi - \frac{1}{2}\log\tilde h_k)_z (\dot f + f_z \dot z_k) \big)(z_k)
\end{aligned}$$

completes the proof. □

7. Holomorphic variation of τ_B

In this section, we prove the following theorem.

Theorem 7.1. *For $\varpi \in T^{1,0}\tilde{\mathcal{H}}_g(1,\dots,1)$ at a point corresponding to a marked Riemann surface X and a holomorphic 1-form Φ on X, and the corresponding $\mu \in \mathcal{H}^{-1,1}(X)$, we have*

$$\begin{aligned}
\partial \log \tau_B^{24}(\varpi) =& \frac{4}{\pi} \lim_{\delta\to 0} \int_{X_\delta} (R_B - R_\Phi)\, \mu\, d^2 z \\
&+ \sum_{p_k\in Z} \big(6\dot f_z + 3(\log\tilde h_k)^{\cdot} + 3(\log\tilde h_k)_z \dot f - (\log\tilde h_k)_z f_z \dot z_k \big)(z_k).
\end{aligned} \tag{7.1}$$

Proof. By the chain rule, first we have

$$\partial \log \tau_B(\varpi) = \sum_{1\le i\le g, 2\le k\le 2g-2} \Big(\frac{\partial \log\tau_B}{\partial A_i}\frac{\partial A_i}{\partial\mu} + \frac{\partial\log\tau_B}{\partial B_i}\frac{\partial B_i}{\partial\mu} + \frac{\partial\log\tau_B}{\partial Z_k}\frac{\partial Z_k}{\partial\mu} \Big),$$

where A_i, B_i, Z_k are the coordinates on $\widetilde{\mathcal{H}}_g(1,\dots,1)$ given in equation (2.2). The holomorphic variation of the coordinates A_i is given by

$$\frac{\partial A_i}{\partial\mu}=\int_{a_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z,$$

and similar equalities hold for B_i, Z_k. Combining these and the defining equations of τ_B in (2.3),

$$\begin{aligned}
&\partial\log\tau_B^{24}(\varpi)\\
=&\frac{2i}{\pi}\Big(-\sum_{i=1}^{g}\int_{b_i}\frac{(R_B-R_\Phi)}{h}\,dz\cdot\big(\int_{a_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\\
&\quad+\sum_{i=1}^{g}\int_{a_i}\frac{(R_B-R_\Phi)}{h}\,dz\cdot\big(\int_{b_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\\
&\quad+\sum_{k=1}^{2g-2}\int_{|z-z_k|=\delta}\frac{(R_B-R_\Phi)}{h}\,dz\cdot\big(\int_{z_1}^{z_k}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\Big).
\end{aligned}\tag{7.2}$$

On the other hand, for the integral on the right hand side of (7.1) we have

$$\begin{aligned}
&\frac{4}{\pi}\int_{X_\delta}(R_B-R_\Phi)\,\mu\,d^2z\\
=&\frac{2i}{\pi}\int_{X_\delta}\frac{(R_B-R_\Phi)}{h}\,dz\wedge\big((\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z\big)\\
=&-\frac{2i}{\pi}\int_{X_\delta}d\Big(\int_{z_1}^{z}(\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z\cdot\frac{(R_B-R_\Phi)}{h}\,dz\Big).
\end{aligned}$$

Here $(\dot h+h_z\dot f+h\dot f_z)\,dz+(h\dot f)_{\bar z}\,d\bar z$ is a globally well-defined 1-form so that its line integral defines a well-defined function. As in the proof of the Riemann's bilinear relation to the last line in the above equalities, we have

$$\begin{aligned}
&\frac{4}{\pi}\int_{X_\delta}(R_B-R_\Phi)\,\mu\,d^2z\\
=&\frac{2i}{\pi}\Big(-\sum_{i=1}^{g}\int_{b_i}\frac{(R_B-R_\Phi)}{h}\,dz\cdot\big(\int_{a_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\\
&\quad+\sum_{i=1}^{g}\int_{a_i}\frac{(R_B-R_\Phi)}{h}\,dz\cdot\big(\int_{b_i}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\\
&\quad+\sum_{k=1}^{2g-2}\int_{|z-z_k|=\delta}\big(\int_{z_1}^{z}(\dot h+h_z\dot f+h\dot f_z)\,dz+h\dot f_{\bar z}\,d\bar z\big)\,\frac{(R_B-R_\Phi)}{h}\,dz\Big).
\end{aligned}\tag{7.3}$$

Near $z_k\in Z$ where we have $\Phi(z)=(z-z_k)\tilde h_k dz$, $\frac{R_\Phi}{h}$ has the following expression

$$\frac{R_\Phi}{h}(z)=-\frac{3}{2}\frac{1}{\tilde h(z_k)}\frac{1}{(z-z_k)^3}+\frac{1}{2}\frac{\tilde h_z(z_k)}{\tilde h^2(z_k)}\frac{1}{(z-z_k)^2}+\Big(3\frac{\tilde h_{zz}(z_k)}{\tilde h^2(z_k)}-\frac{\tilde h_z^2(z_k)}{\tilde h^3(z_k)}\Big)\frac{1}{z-z_k}+\cdots$$

where $\tilde{h} = \tilde{h}_k$. Now comparing (7.2) with (7.3), in order to complete the proof, it is sufficient to show that

$$\begin{aligned}\lim_{\delta\to 0} \frac{2i}{\pi} \sum_{p_k\in Z} \int_{|z-z_k|=\delta} \Big(\int_{z_1}^{z} (\dot{h} + h_z\dot{f} + h\dot{f}_z)\, dz + h\dot{f}_{\bar z}\, d\bar z \Big) \big(\frac{R_\Phi}{h}\big)_s\, dz \\ = \sum_{p_k\in Z} \big(6\dot{f}_z + 3(\log\tilde{h}_k)^{\cdot} + 3(\log\tilde{h}_k)_z\dot{f} - (\log\tilde{h}_k)_z f_z \dot{z}_k\big)(z_k)\end{aligned} \tag{7.4}$$

where $(\frac{R_\Phi}{h})_s := \big(-\frac{3}{2}\frac{1}{\tilde{h}(z_k)}\frac{1}{(z-z_k)^3} + \frac{1}{2}\frac{\tilde{h}_z(z_k)}{\tilde{h}^2(z_k)}\frac{1}{(z-z_k)^2}\big)$ with $\tilde{h} = \tilde{h}_k$ near z_k. By some elementary computations, we have

$$\begin{aligned}&\int_{|z-z_k|=\delta} \Big(\int_{z_1}^{z} (\dot{h} + h_z\dot{f} + h\dot{f}_z)\, dz + h\dot{f}_{\bar z}\, d\bar z \Big) \frac{1}{(z-z_k)^2}\, dz \\ &\qquad = 2\pi i(\dot{h} + h_z\dot{f} + h\dot{f}_z)(z_k) + O(\delta) = -2\pi i(\tilde{h}_k f_z)(z_k)\dot{z}_k + O(\delta),\end{aligned}$$

$$\begin{aligned}&\int_{|z-z_k|=\delta} \Big(\int_{z_1}^{z} (\dot{h} + h_z\dot{f} + h\dot{f}_z)\, dz + h\dot{f}_{\bar z}\, d\bar z \Big) \frac{1}{(z-z_k)^3}\, dz \\ &= \pi i(\dot{h}_z + h_{zz}\dot{f} + 2h_z\dot{f}_z + h\dot{f}_{zz})(z_k) + O(\delta) = \pi i\big(2\dot{f}_z\tilde{h}_k + \dot{\tilde{h}}_k + \tilde{h}_{kz}(\dot{f} - f_z\dot{z}_k)\big)(z_k) + O(\delta).\end{aligned}$$

The equality (7.4) follows from these and this completes the proof. □

8. Proof of Theorem 1.1 for $\tilde{\mathcal{H}}_g(1,\ldots,1)$

In this section we collect the formulae proved in the previous sections to prove Theorem 1.1 for $\tilde{\mathcal{H}}_g(1,\ldots,1)$. For this, first we recall a property of the Schwarzian derivative:

$$\mathcal{S}(h_1\circ h_2) = \mathcal{S}(h_1)\circ h_2 (h_2')^2 + \mathcal{S}(h_2). \tag{8.1}$$

Let $\pi_F : H^2 \to X$ and $\pi_S : \Omega \to X$ denote the Fuchsian and Schottky uniformization maps respectively. Then the universal covering map $J : H^2 \to \Omega$ satisfies $\pi_F = \pi_S \circ J$. (In case $g = 1$, replace H^2 with $\mathbb{C}$.) Applying this to the composition of multi-valued functions $J^{-1} = \pi_F^{-1}\circ\pi_S$, we obtain

$$\mathcal{S}(J^{-1}) = \mathcal{S}(\pi_F^{-1})\circ\pi_S(\pi_S')^2 - \mathcal{S}(\pi_S^{-1})\circ\pi_S(\pi_S')^2.$$

Similarly applying (8.1) to the composition of multi-valued functions $h_\Phi = (\int^z\Phi)\circ\pi_S$ for a local coordinate z over X, we obtain

$$\mathcal{S}(h_\Phi) = \mathcal{S}(\int^z\Phi)\circ\pi_S(\pi_S')^2 - \mathcal{S}(\pi_S^{-1})\circ\pi_S(\pi_S')^2.$$

Let us recall that $\mathcal{S}(\pi_F^{-1})$, $\mathcal{S}(\pi_S^{-1})$, $\mathcal{S}(\int^z\Phi)$ define the projective connections R_F, R_S, R_Φ over X respectively.

Given a point $u_0 \in U \subset \tilde{\mathcal{H}}_g^*(1,\ldots,1)$, with U a contractible open set, and a tangent vector $\varpi \in T^{1,0}U$ at u_0, we have a corresponding $\mu \in \mathcal{H}^{-1,1}(X)$, family of deformations $f_{w\mu}$ and holomorphic family of holomorphic 1-forms $\Phi(w)$. For this family, by Theorem 5.14 and

Corollary 5.15, we have

$$\begin{aligned}\partial(4\pi\overline{\mathbb{CS}})(\varpi) = &-\frac{2}{\pi}\lim_{\delta\to 0}\int_{X_\delta}\big((R_F - R_S) - (R_\Phi - R_S)\big)\,\mu\, d^2z \\ &-\sum_{p_k\in Z}\big(3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z \dot{z}_k\big)(z_k).\end{aligned}\tag{8.2}$$

$$\begin{aligned}\partial(4\pi\mathbb{CS})(\varpi) = &-\frac{2}{\pi}\lim_{\delta\to 0}\int_{X_\delta}\big((R_F - R_S) + (R_\Phi - R_S)\big)\,\mu\, d^2z \\ &+\sum_{p_k\in Z}\big(3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z \dot{z}_k\big)(z_k).\end{aligned}\tag{8.3}$$

By Theorem 6.1, we also have

$$\begin{aligned}\partial(\frac{1}{\pi}I)(\varpi) = &\frac{2}{\pi}\lim_{\delta\to 0}\int_{X_\delta}(R_F - R_\Phi)\,\mu\, d^2z \\ &+\sum_{p_k\in Z}\big(3\dot{f}_z + \frac{3}{2}(\log\tilde{h}_k)^{\cdot} + \frac{3}{2}(\log\tilde{h}_k)_z\dot{f} - \frac{1}{2}(\log\tilde{h}_k)_z f_z \dot{z}_k\big)(z_k),\end{aligned}\tag{8.4}$$

where we have lifted the function I from $\mathcal{H}_g(1,\ldots,1)$ to $U\subset\tilde{\mathcal{H}}^*_g(1,\ldots,1)$. From (8.2) and (8.4), it follows that $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)$ is a holomorphic function over U. It is known that

$$\partial(\log F^{24})(\varpi) = \partial(\log F^{24})(\mu) = \frac{4}{\pi}\int_X (R_B - R_S)\,\mu\, d^2z \tag{8.5}$$

from [15], [19]. Here we also have lifted the function F from $\mathfrak{T}_g$ to $U\subset\tilde{\mathcal{H}}^*_g(1,\ldots,1)$. Combining (8.3), (8.4), and (8.5), the holomorphic variation of the holomorphic function $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)\,F^{24}$ is given by

$$\begin{aligned}&\partial\log\big(\,\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)\,F^{24}\,\big)(\varpi) \\ =&\frac{4}{\pi}\lim_{\delta\to 0}\int_{D_\delta}(R_B - R_\Phi)\,\mu\, d^2z \\ &+\sum_{p_k\in Z}\big(6\dot{f}_z + 3(\log\tilde{h}_k)^{\cdot} + 3(\log\tilde{h}_k)_z\dot{f} - (\log\tilde{h}_k)_z f_z \dot{z}_k\big)(z_k).\end{aligned}\tag{8.6}$$

By Theorem 7.1 and equation (8.6), the two functions τ_B^{24} (lifted to U) and $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)F^{24}$ have the same holomorphic variation for any holomorphic tangent vector ϖ. Consequently, the liftings of τ_B^{24} and $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)F^{24}$ to any connected component of $\tilde{\mathcal{H}}^*_g(1,\ldots,1)$ are equal up to a multiplicative constant. But the holomorphic function τ_B^{24} descends to $\tilde{\mathcal{H}}_g(1,\ldots,1)$, hence the holomorphic function $\exp(4\pi\mathbb{CS}+\frac{1}{\pi}I)F^{24}$ descends too. This proves Theorem 1.1. The constant c appearing in the theorem depends on our choice of a connected component of $\tilde{\mathcal{H}}^*_g(1,\ldots,1)$.

9. Proof of Theorem 1.1 for $\tilde{H}_{g,n}(1,\ldots,1)$

A point in $\tilde{H}_{g,n}(1,\ldots,1)$ corresponds to an equivalence class of a compact Riemann surface X of genus g, together with a meromorphic function $\lambda : X\to\mathbb{CP}^1$. The differential $d\lambda$ is meromorphic, with m simple zeros at the ramification points $p_1,\ldots p_m\in X$ of λ, and with n

double poles at the preimages of infinity $q_1, \dots, q_n$. It has residue 0 at each pole. The proof of Theorem 1.1 given above applies, with $d\lambda$ playing the role of Φ, with some modifications due to the poles of $d\lambda$. We outline these modifications in this section. Note that we will carry over all definitions and notations from before, with any changes being noted below.

9.1. **Construction of framing.** Recall that Z is the zero set of $d\lambda$ on X; denote also by P the set of poles of $d\lambda$ on X. The framing on $X \setminus (Z \cup P)$ is constructed from $d\lambda$ as before; it will now have a singularity of index 2 at each point in P. Let $z_{j\alpha}$ denote the co-ordinate of a pole of $d\lambda$ in a patch U_α. Then $h_\alpha = (z_\alpha - z_{j\alpha})^{-2}\tilde{h}_{j\alpha}$, where $\tilde{h}_{j\alpha}$ is holomorphic and non-zero on U_α. Note that

$$(\log \tilde{h}_{j\alpha})_z(z_\alpha(q_j)) = 0 \quad \text{for} \quad q_j \in P. \tag{9.1}$$

As before, we define $\tilde{\theta}$ by $e^{i\tilde{\theta}_\alpha} := \tilde{h}_\alpha/|\tilde{h}_\alpha|$. We now define the co-framing $(\tilde{\omega}_2, \tilde{\omega}_3)$ and framing $(\tilde{f}_2, \tilde{f}_3)$ as before, but by means of $e^{\frac{1}{2}\phi_\alpha - i\tilde{\theta}_\alpha} dz_\alpha$ rather than $e^{\frac{1}{2}(\phi_\alpha + i\tilde{\theta}_\alpha)} dz_\alpha$.

For each point $q_j \in P$, the admissible extension of the framing on X will have an additional singular curve with both endpoints at q_j. We denote the set of these curves by $\mathcal{L}^2_P$. (Note that in this notation, $\mathcal{L}^2$ and $\mathcal{L}^2_P$ are disjoint.) We require the reference framing κ on each component of this curve to satisfy $r^{-1}\kappa \to (\tilde{f}_2, \tilde{f}_3)$ at the outgoing endpoint, and $r^{-1}\kappa \to (\tilde{f}_2, -\tilde{f}_3)$ at the incoming endpoint, as $r \to 0$ (identifying framings on $\partial\overline{M}$ and X as in subsection 3.3). The proof of the existence of admissible extensions (Theorem 3.3) must be modified as follows: for the singular curve with endpoints at a pole q_j, choose a small neighborhood of its intersection with the level surface X^{a_1}. Take the subset of $N_{[0,a_1)}$ consisting of all geodesics connecting points of this neighborhood to q_j. The framing on $N_{[0,a_1)}$ will be defined by parallel translation outside of this neighborhood, as before. Inside the neighborhood, we pick any framing which matches smoothly on the boundary, which has e_1 orthogonal to the level surface, and which has an index 1 singularity at each of the two components of the singular curve. The framing is then extended to the rest of M as before. The resulting framing is an admissible extension of the boundary framing given by $d\lambda$.

9.2. **Variation of the invariant $\mathbb{CS}$.** The fact that the singularity of the framing around $\mathcal{L}^2_P$ is index 1 means that the corresponding boundary contributions computed in Section 4 appear with opposite sign to those from the components of $\mathcal{L}^2$ with endpoints at the zeroes of $d\lambda$. Since the contribution from $\mathcal{L}^2_P$ in the definition 4.2 also appears with opposite sign, the results of Section 4 hold without change.

In the remaining sections, we have the following changes. We have the following new contribution from $\mathcal{L}^2_P$ in Proposition 5.8,

$$-\frac{1}{2\pi} \sum_{y \in \partial\mathcal{L}^2_P} \sigma^*(\theta_1 + i\theta_{23})\big|_y$$

which leads to the new contribution in Proposition 5.12,

$$
\begin{aligned}
&\Big(-\frac{1}{2\pi} \sum_{y\in(\partial\mathcal{L}^2_P\cap D)} \sigma^*(\theta_1 + i\theta_{23})\big|_y \Big)(\frac{\partial}{\partial w}) \\
&= -\frac{1}{2\pi} \sum_{z_j\in P} \big((\dot\phi + 2i\tilde\theta^{\cdot}) + (\phi + 2i\tilde\theta)_z \dot f + (\phi + 2i\tilde\theta)_z f_z \dot z_j \big)(z_j) \\
&= -\frac{1}{2\pi} \sum_{z_j\in P} \big(-\dot f_z + (\log\tilde h_j)^{\cdot} + \phi_z f_z \dot z_j \big)(z_j)
\end{aligned}
\tag{9.2}
$$

where we used equation (9.1). We also have the following new contribution from the set P in Proposition 5.13,

$$
\begin{aligned}
&\frac{i}{8\pi^2} \lim_{\delta\to 0} \sum_{z_j\in P} \int_{|z-z_j|=\delta} (\phi - 2i\theta)_z \big((\phi - 2i\theta)\circ f \big)^{\cdot} dz \\
=&\frac{i}{8\pi^2} \lim_{\delta\to 0} \sum_{z_j\in P} \Big(\int_{|z-z_j|=\delta} (\frac{2}{z-z_j} + (\phi - \log\tilde h_j)_z) \big(2\,\frac{\dot f(z) - \dot f(z_j) - f_z(z_j)\dot z_j}{z-z_j} \big)\, dz \\
&\qquad + \int_{|z-z_j|=\delta} (\frac{2}{z-z_j} + (\phi - \log\tilde h_j)_z) \big((\phi - \log\tilde h_j)\circ f \big)^{\cdot}(z) \big)\, dz \Big) \\
=& -\frac{1}{4\pi} \sum_{z_j\in P} \big(4\dot f_z(z_j) - 2(\phi - \log\tilde h_j)_z(z_j) f_z(z_j)\dot z_j + 2((\phi - \log\tilde h_j)\circ f)^{\cdot}(z_j) \big) \\
=& -\frac{1}{2\pi} \sum_{z_j\in P} \big(\dot f_z - (\log\tilde h_j)^{\cdot} - \phi_z f_z \dot z_j \big)(z_j).
\end{aligned}
\tag{9.3}
$$

Since the right sides of (9.2) and (9.3) cancel each other, Theorem 5.14 holds without modification.

For $I(X, d\lambda)$, we have to modify its definition by adding terms from the set P:

$$
\begin{aligned}
I(X, d\lambda) = \lim_{\delta\to 0} \Big(&\int_{X_\delta} |\psi_z|^2\, d^2z \\
&+ \frac{i}{2} \sum_{p_k\in Z} \int_{S_\delta(z_k)} \frac{(\phi - 2\log|h|)(z)}{\bar z - \bar z_k} d\bar z - i \sum_{q_j\in P} \int_{S_\delta(z_j)} \frac{(\phi - 2\log|h|)(z)}{\bar z - \bar z_j} d\bar z \Big) \\
&- \pi \sum_{p_k\in Z} (\phi - \log|\tilde h_k|)(z_k) + 2\pi \sum_{q_j\in P} (\phi + 2\log|\tilde h_j|)(z_j).
\end{aligned}
\tag{9.4}
$$

Here z, in the integral around p_k and q_j, represents a local coordinate near p_k, q_j with $z_k = z(p_k)$, $z_j = z(q_j)$. The holomorphic variation formula of this I can be computed as in the proof of Theorem 6.1 with some new contributions from the set P. First, we have the following new contribution in (6.6),

$$
\begin{aligned}
&-\frac{i}{2}\sum_{q_j\in P}\Big(\int_{|z-z_j|=\delta_0} 4\,\big(\frac{\dot f(z)-\dot f(z_j)-f_z(z_j)\dot z_j}{z-z_j}\big)\big(\frac{1}{z-z_j}dz+\frac{1}{\bar z-\bar z_j}d\bar z\big)\\
&\qquad\qquad +2\,\big(\frac{\dot f(z)-\dot f(z_j)-f_z(z_j)\dot z_j}{z-z_j}\big)\big((\phi-\log\tilde h_j)_z dz+(\phi-\log\bar{\tilde h}_j)_{\bar z}d\bar z\big)\\
&\qquad\qquad +2\,((\phi-\log\tilde h_j)\circ f)^{\cdot}(z)\,\big(\frac{1}{z-z_j}dz+\frac{1}{\bar z-\bar z_j}d\bar z\big)\Big)+O(\delta_0)\\
&= -2\pi\sum_{q_j\in P}(\phi_z f_z)(z_j)\dot z_j+O(\delta_0).
\end{aligned}
\tag{9.5}
$$

Secondly, we have the following new contribution to (6.11),

$$
\partial\Big(-i\sum_{q_j\in P}\int_{S_{\delta_0}(z_j)}\frac{(\phi-2\log|h|)}{\bar z-\bar z_{j,w}}\,d\bar z\Big)(\varpi)=2\pi\sum_{q_j\in P}\big(\dot f_z-(\log\tilde h_j)^{\cdot}\big)(z_j)+O(\delta_0).
\tag{9.6}
$$

For the holomorphic variation of the last term in $I(X,d\lambda)$, we have

$$
\partial\Big(2\pi\sum_{q_j\in P}(\phi+2\log|\tilde h_j|)(z_j)\Big)(\varpi)=2\pi\sum_{q_j\in P}\big(-\dot f_z+(\log\tilde h_j)^{\cdot}+\phi_z f_z\dot z_j\big)(z_j).
\tag{9.7}
$$

Hence, the new contributions from (9.5), (9.6), and (9.7) cancel each other, and Theorem 6.1 holds without modification.

The holomorphic variation of $\log\tau_B^{24}$ is given by the same formula as in Theorem 7.1. The same proof as in Theorem 7.1 also works for this case since the holomorphic variations of $\int_{a_i}d\lambda$ and $\int_{b_i}d\lambda$ vanish. The only possible difference in the proof is the contributions of residues of $\frac{R_{d\lambda}}{\lambda_z}$ at $q_j\in P$ in (7.2) and (7.3). But one can see that this is regular at $q_j\in P$ and there is no contribution from the set P.

Finally, in Section 8, we showed that Theorem 1.1 follows from Theorems 5.14, 6.1, and 7.1. Since these Theorems still hold in this case, Theorem 1.1 follows in this case as well.

References

[1] L. V. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. (2) **74** (1961) 171–191.

[2] S. S. Chern and J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974), 48–69.

[3] B. Dubrovin, *Geometry of 2 D topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math. **1620** (1996), 120–348.

[4] B. Dubrovin and Y. Zhang, *Bi-Hamiltonian hierarchies in 2 D topological field theory at one-loop approximation*, Comm. Math. Phys. **198** (1998), no. 2, 311–361.

[5] C. R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999), Rend. Circ. Mat. Palermo (2) Suppl. **63** (2000), 31–42.

[6] C. Guillarmou, S. Moroianu and J. Park, *Eta invariant and Selberg zeta function of odd type over convex co-compact hyperbolic manifolds*, Adv. Math. **225** (2010), no. 5, 2464–2516.

[7] A. Kokotov and D. Korotkin, *Tau-functions on Hurwitz spaces*, Math. Phys. Anal. Geom. **7** (2004), no. 1, 47–96.

[8] A. Kokotov and D. Korotkin, *On G -function of Frobenius manifolds related to Hurwitz spaces*, Int. Math. Res. Not. (2004), no. 7, 343–360.

[9] A. Kokotov and D. Korotkin, *Isomonodromic tau-function of Hurwitz Frobenius manifolds and its applications*, Int. Math. Res. Not. (2006), 1–34.

[10] A. Kokotov and D. Korotkin, *Tau-functions on spaces of abelian differentials and higher genus generalizations of Ray-Singer formula*, J. Differential Geom. **82** (2009), no. 1, 35–100.

[11] M. Kontsevich and A. Zorich, *Connected components of the moduli spaces of Abelian differentials with prescribed singularities*, Invent. Math. **153** (2003), no. 3, 631–678.

[12] K. Krasnov, *Holography and Riemann surfaces*, Adv. Theor. Math. Phys. **4** (2000), no. 4, 929–979.

[13] K. Krasnov and J. M. Schlenker, *On the renormalized volume of hyperbolic* 3*-manifolds*, Comm. Math. Phys. **279** (2008), no. 3, 637–668.

[14] Y. Manin, *Three-dimensional hyperbolic geometry as* ∞*-adic Arakelov geometry*, Invent. Math. **104** (1991), no. 2, 223–243.

[15] A. McIntyre and L. A. Takhtajan, *Holomorphic factorization of determinants of Laplacians on Riemann surfaces and a higher genus generalization of Kronecker's first limit formula*, Geom. Funct. Anal. **16** (2006), no. 6, 1291–1323.

[16] R. Meyerhoff, *The Chern-Simons invariant for hyperbolic* 3*-manifolds*, Thesis, Princeton: Princeton University Press, N. J., 1981.

[17] L. A. Takhtajan and L. P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography*, Comm. Math. Phys. **239** (2003), no. 1-2, 183–240.

[18] T. Yoshida, *The* η*-invariant of hyperbolic* 3*-manifolds*, Invent. Math. **81** (1985), no. 3, 473–514.

[19] P. G. Zograf, *Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces*, Leningrad Math. J. **1** (1990), no. 4, 941–965.

[20] P. G. Zograf, *Determinants of Laplacians, Liouville action, and an analogue of the Dedekind* η*-function on Teichmüller space*, unpublished manuscript (1997).

[21] P. G. Zograf and L. A. Takhtadzhyan, *On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmuller and Schottky spaces*, Math. USSR-Sb. **60** (1988), no. 2, 297–313.

Bennington College, 1 College Drive, Bennington, Vermont 05201, U.S.A.
E-mail address: amcintyre@bennington.edu

School of Mathematics, Korea Institute for Advanced Study, 207-43, Hoegiro 85, Dongdaemun-gu, Seoul 130-722, Korea
E-mail address: jinsung@kias.re.kr